Kurdistan Regional Government
Ministry of Higher Education and
Scientific Research
University of Sulaimani
Faculty of Science and Science Education
School of Science Education
Department of Mathematics

A Study of Fractional Lacunary Interpolations by Spline with Applications

A thesis
submitted to the Council of
Faculty of Science and Science Education
School of Science Education at the University of Sulaimani
in partial fulfillment of the requirements for the degree of
Master of Science in Mathematics
(Numerical Analysis)

By

Pshtiwan Othman Muhammad

B.Sc. Mathematics (2012), University of Sulaimani

Supervised by

Dr. Faraidun Kadir Hamasalh

Assistant Professor

Kharmanan 25 2715 August 16 2015

Supervisor Certification

I certify that the preparation of thesis “A Study of Fractional Lacunary Interpolations by Spline with Applications” conducted by Pshtiwan Othman Muhammad, was prepared under my supervision at the School of Science Education, Faculty of Science and Science Education at the University of Sulaimani, as partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

Signature:

Name: Dr. Faraidun Kadir Hamasalh

Title: Assistant Professor

Date: / / 2015

In view of the available recommendation, I forward this thesis for discussion by the examining committee.

Signature:

Name: Dr. Arkan Nawzad Mustafa

Title: Lecturer

Date: / / 2015

Linguistic Evaluation Certification

I hereby certify that this thesis titled "A Study of Fractional Lacunary Interpolations by Spline with Applications", prepared by Pshtiwan Othman Muhammad, has been read and checked and after indicating all the grammatical and spelling mistakes; the thesis was given again to the candidate to make the adequate corrections. After the second reading, I found that the candidate corrected the indicated mistakes. Therefore, I certify that this thesis is free from mistakes.

Signature:

Name: Jutyar Omar Salh

Position: School of Language and Humanities-English Department

Date: / / 2015

Dedication

This thesis is dedicated to the memory of my parents.

Acknowledgments

I would like to bow my head before Allah Almighty, the Most Gracious and the Most Merciful, whose benediction bestowed upon me, provided me with sufficient opportunity and enabled me to undertake and execute this research work.

Throughout the completion of this work, I have been supported and guided by several people. I would like to take this opportunity to express my gratitude to all those people. My deepest and sincere gratitude and appreciation goes to my supervisor Assistant Professor Dr. Faraidun K. Hamasalh for his guidance at each stage of this work. His patience, encouragement and support have been very valuable in the completion of this thesis. His advises were always stimulating and helpful when I was facing difficulties in my research. His mission of producing high-quality work will always help me grow and expand my thinking. I am also thankful to my teacher Dilan Faraidun at the School of Sciences Education for his encouragement during this research work.

I wish to record my deepest obligations to friends, parents, brothers and sisters for their unfailing support and many sacrifices at every stage during these years. I am also grateful to all those who shared their knowledge with me and made suggestions reflected in this thesis.

Pshtiwan Othman

List of Symbols

Symbols	Descriptions
$\Gamma(a)$	Gamma function of a
$B(a, b)$	Beta function of a and b
$\mathbb{R}$	Set of real numbers
$E_\alpha(x)$	One-parameter Mittag-Leffler function of x
$E_{\alpha,\beta}(x)$	Two-parameters Mittag-Leffler function of x
$I_a^\alpha f(x)$	Riemann-Liouville fractional integral of order $\alpha > 0$ of the function $f(x)$
$D_a^\alpha f(x)$	Riemann-Liouville fractional derivative of order $\alpha > 0$ of the function $f(x)$
${}_cD_a^\alpha f(x)$	Caputo fractional derivative of order $\alpha > 0$ of the function $f(x)$
${}^GD^\alpha y(x)$	Grünwald-Letnikov derivative of the function $y(x)$
$\binom{n}{k}$	The kth binomial coefficient of order n
$C^\infty(X)$	Set of all functions having derivatives of all orders on X
Π_n	Set of all polynomials of degree n or less
$\omega_{m\alpha}(h)$	The modulus of continuity of $D^{m\alpha}y(x)$
$\|\|\mathbf{x}\|\|$	The l_∞ norm of the vector $\mathbf{x}$
$\|\|A\|\|$	The l_∞ norm of the matrix A
$i(j)n$	$i, i + j, i + 2j, ..., n.$

Published and Submitted papers

Certain aspects of this thesis are based on the following published/submitted papers:

Published Journal papers

[1] Hamasalh F. K. and Muhammad P. O., *Analysis of Fractional Splines Interpolation and Optimal Error Bounds*, American Journal of Numerical Analysis, 2015, Vol. 3, No. 1, 30-35.

[2] Hamasalh F. K. and Muhammad P. O., *Generalized Quartic Fractional Spline Interpolation with Applications*, Int. J. Open Problems Compt. Math, Vol. 8, No. 1 (2015), 67-80.

[3] Hamasalh F. K. and Muhammad P. O., *An Algorithm for The Fractional Spline Approximation Function with Applications*, second scientific conference of Garmian University, No. 134, 2015.

Submitted Journal paper

[4] Hamasalh F. K. and Muhammad P. O., *Numerical Solution of Fractional Differential Equations by using Fractional Spline Functions*, accepted for publication.

Abstract

The main aim of this thesis was to present some derivations and studying fractional lacunary data using spline functions and then solving some examples on fractional differential equations numerically.

Firstly, a new fractional spline function of polynomial form with the idea of the lacunary interpolation is considered to find approximate solution for fractional differential equations (FDEs). The proposed method is applicable for $\alpha \in (0, 1]$, where α denotes the order of the fractional derivative in the Caputo sense. Convergence analysis of the method is considered. Some illustrative examples are presented and the obtained results reveal that the proposed technique is very effective, convenient and quite accurate to such considered problems.

Finally, we present a formulation and a study of three interpolatory fractional splines in the class of $m\alpha$, $m = 2, 4, 6$, $\alpha = 0.5$. We extend fractional splines function with uniform knots to approximate the solution of fractional equations. The developed spline method is used to analyse convergence fractional order derivatives and estimating error bounds. We propose spline fractional method to solve fractional differentiation equations.

Contents

Dedication v

Acknowledgments vi

List of Symbols vii

Published and Submitted papers viii

Abstract ix

List of Figures xiii

List of Tables xiv

1 Introduction 2

1.1 Historical Note and Literature Survey 2

1.2 Special Functions 4

1.2.1 Beta Function 4

1.2.2 Gamma Function 4

1.2.3 Some Properties of the Gamma Function 5

1.2.4 Mittag-Leffler Function 6

1.3 The Fractional Derivatives and Integrals 9

1.3.1 The Riemann-Liouville Fractional Differintegral Operator 9

1.3.2 The Caputo Fractional Differential Operator 11
1.3.3 The Grünwald-Letnikov Derivative 14
1.4 Interpolation by Polynomial Spline Functions 15
1.4.1 Lacunary Interpolation 15
1.4.2 Modulus of Continuity 16
1.4.3 Some Theorems of Error Bounds 16
1.4.4 Advantages of Spline Functions 17

2 Numerical Solution of FDEs by using Fractional Lacunary Spline Functions 20
2.1 Introduction 20
2.2 Generalized quartic fractional spline interpolation with applications 21
2.2.1 Descriptions 21
2.2.2 Existence and Uniqueness 21
2.2.3 Error Bounds 23
2.2.4 Numerical Illustrations 31

3 Analysis of Fractional Spline Interpolation 36
3.1 Introduction 36
3.2 Analysis of Fractional Splines Interpolation–$\left(0, \frac{1}{2}\right)$ Case and Optimal Error bounds 37
3.2.1 Spline of Degree 2α (Existence and Uniqueness) 37
3.2.2 Error Bounds for the Fractional Spline 39
3.2.3 Spline of Degree 4α Case (Existence and Uniqueness) 40
3.2.4 Error Bounds for the Fractional Spline of Degree 4α Case 41
3.2.5 Spline of Degree 6α Case (Existence and Uniqueness) 42
3.2.6 Error Bounds for the Fractional Spline of Degree 6α Case 43
3.2.7 Algorithms 44
3.2.8 Illustrations 44

3.3 General algorithms for the Fractional Spline Approximation Function with Applications 46
3.3.1 Existence and Uniqueness 47
3.3.2 Error Estimations 49
3.3.3 Algorithms 53
3.3.4 Numerical Illustrations 53

4 Conclusions and Future Works **57**
4.1 Conclusions 57
4.2 Future Works 58

Bibliography **60**

List of Figures

1.1 Graph of the Gamma function $\Gamma(x)$ in a real domain. 5

1.2 Mittag-Leffler function (1.6) for various parameters 8

3.1 Exact and approximate solutions of Example 3.3 with $h = 0.2$. 46

3.2 Exact and approximate solutions of Example 3.6 with $h = 0.25$. 55

List of Tables

2.1 Maximal absolute errors in case 1 where $\alpha = 0.5$ for Example 2.1. 31

2.2 Maximal absolute errors in case 2 where $\alpha = 0.5$ for Example 2.1. 31

2.3 Maximal absolute errors in case 1 where $\alpha = 0.8$ for Example 2.1. 32

2.4 Maximal absolute errors in case 2 where $\alpha = 0.8$ for Example 2.1. 32

2.5 Maximal absolute error in case 1 where $\alpha = 0.5$ for Example 2.2. 32

2.6 Maximal absolute error in case 2 where $\alpha = 0.5$ for Example 2.2. 33

2.7 Maximal absolute errors in case 1 where $\alpha = 0.8$ for Example 2.2. 33

2.8 Maximal absolute errors in case 2 where $\alpha = 0.8$ for Example 2.2. 33

2.9 Error bounds for Example 2.3 when $\alpha = 0.5$. 34

2.10 Error bounds for Example 2.3 when $\alpha = 0.8$. 34

3.1 The observed maximum errors 45

3.2 The observed maximum errors 45

3.3 Exact, approximate and absolute error 46

3.4 Exact, approximate and absolute error 54

3.5 Maximum absolute error for Example 3.5 54

3.6 Exact, approximate and absolute error 55

Chapter One

Introduction

Chapter 1

Introduction

1.1 Historical Note and Literature Survey

There are many theoretical results on existence, uniqueness, and properties of solutions of ordinary and partial differential equations, usually only the simplest specific problems can be solved explicitly, especially when the nonlinear terms are involved and we usually construct approximate solutions. Since only limited classes of the equations are solved by analytical means, numerical solution of these differential equations is of practical importance. Polynomials have long been the functions most widely used to approximate other functions mainly because of their simple mathematical properties. However, it is well-known that polynomials of high degree tend to oscillate strongly and in many cases they are liable to produce very poor approximations. With a spline function (spline function is a numerical function that is piecewise-defined by polynomial functions, and which possesses a sufficiently high degree of smoothness at the places where the polynomial pieces connect (which are known as knots) [2]), low degree and hence weakly oscillating polynomials combined in such a way as to obtain a function which is as smooth as possible in the sense that it has maximal continuity without being globally a polynomial. Spline functions can be integrated and differentiated due to being piecewise polynomials and can be easily stored and implemented on digital computers. Thus, spline functions are adapted to numerical methods to get the solution of the differential equations. Numerical methods with spline functions in getting the approximate solution of the differential equations lead to band matrices which are solvable easily with algorithms having low cost of computation [2, 31].

Many authors [17, 20, 21, 50] presented several local methods for solving lacunary interpolation problems using piecewise polynomials with certain continuity properties. Moreover, they have studied the use of splines to solve the lacunary interpolation problems. All of these methods are global and require the solution of a large system of equations.

The fractional calculus started from some speculations of G.W. Leibniz (1695, 1697) and L. Euler (1730), and it has been developed progressively up to now. A list of mathematicians, who have provided important contributions up to the middle of the twentieth century, include P.S. Laplace (1812), S. F. Lacroix (1819), J. B. J. Fourier (1822), N. H. Abel (1823-1826), J. Liouville (1832-1873), B. Riemann (1847), H. Holmgren (1865-1867), A. K. Grunwald (1867-1872), A. V. Letnikov (1868-1872), H. Laurent (1884), P. A. Nekrassov (1888), A. Krug (1890), J. Hadamard (1892), O. Heaviside (1892-1912), S. Pincherle (1902), G. H. Hardy and J. E. Littlewood (1917-1928), H. Weyl (1917), P. Lévy (1923), A. Marchaud (1927), H. T. Davis (1924-1936), E. L. Post (1930), A. Zygmund (1935-1945), E. R. Love (1938-1996), A. Erdelyi (1939-1965), H. Kober (1940), D. V. Widder (1941), M. Riesz (1949), W. Feller (1952).

Only since the seventies has fractional calculus been the object of specialized conferences and treatises. For the first conference, the merit is due to B. Ross who, shortly after his Ph.D. dissertation on fractional calculus, organized the First Conference on Fractional Calculus and its Applications at the University of New Haven in June 1974, and edited the proceedings [47]. For the first monograph the merit is ascribed to K. B. Oldham and J. Spanier who, after a joint collaboration begun in 1968, published a book devoted to fractional calculus in 1974 [37].

In recent years considerable interest in fractional calculus has been stimulated by the applications it finds in different areas of applied sciences like physics and engineering, possibly including fractal phenomena. Nowadays, the fractional calculus attracts many scientists and engineers. There are several applications of this mathematical phenomenon in mechanics, physics, chemistry, control theory and so on (Caponetto et al., 2010 [8]; Magin, 2006 [30]; Monje et al., 2010 [35]; Oldham and Spanier, 1974 [37]; Oustaloup, 1995 [41]; Podlubny, 1999 [5]).

1.2 Special Functions

In this section, the most important functions used in fractional calculus are mentioned. Furthermore, we have nice examples for a successful extension of the scope of functions e.g. from integer to real values. We begin with three stories of success: we present the gamma, beta and Mittag-Leffler functions, which turn out to be well established extensions of the factorial and the exponential function. These functions play an important role for practical applications of the fractional calculus.

1.2.1 Beta Function

The Beta function is very important for the computation of the fractional derivatives of the power function. It is defined for $\{p, q \in \mathbb{C},\ Re(p) > 0,\ Re(q) > 0\}$ to be [3, 42]:

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$$

Beta function is also called the *First Eulerian Integral.*

1.2.1.1 Some Properties of the Beta Function

(i) **Symmetry of Beta Function**. $B(p,q) = B(q,p)$ (see [42], p. 365).

(ii) If p, q are positive integers, then $B(p,q) = \frac{(p-1)!(q-1)!}{(p+q-1)!}$ (see [42], p. 369).

1.2.2 Gamma Function

In the integer-order calculus the factorial plays an important role because it is one of the most fundamental combinatorial tools. The Gamma function has the same importance in the fractional-order calculus and the gamma function is defined for $\{z \in \mathbb{C},\ z \neq 0, -1, -2, ...\}$ to be [3]:

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx \tag{1.1}$$

Gamma function is also called the *Second Eulerian Integral*.

In view of the Gauss expression (see [42], p. 371), we attain the fact that the gamma function is defined for all $z \in \mathbb{C} - \{0, -1, -2, ...\}$. Moreover, in the sense of complex analysis the negative integers are simple poles of $\Gamma(z)$. For a better understanding the graph of $\Gamma(x)$ for real values of x is given in Figure 1.1 [26].

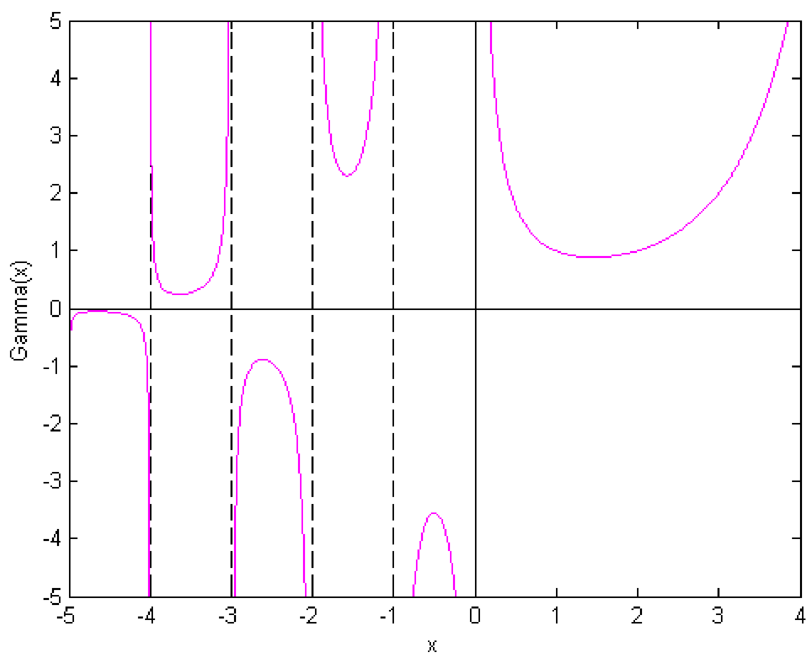


Figure 1.1: Graph of the Gamma function $\Gamma(x)$ in a real domain.

1.2.3 Some Properties of the Gamma Function

We have some major properties of the gamma function which are beneficial to our works, and they are given as:

(i) **Recurrence Formula for Gamma Function** $\Gamma(n)$ (see [42], p. 372).

$$\Gamma(n) = (n-1)\Gamma(n-1), \quad \text{when } n > 1. \tag{1.2}$$

For example: $\Gamma(1) = 1$, and using (1.1) we obtain for $n \in \mathbb{N}$ (i.e., $n = 1, 2, 3, ...$):

$$\Gamma(2) = 1.\Gamma(1) = 1 = 1!,$$

$$\Gamma(3) = 2.\Gamma(2) = 2.1! = 2!,$$

$$\Gamma(4) = 3.\Gamma(3) = 3.2! = 3!,$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\Gamma(n+1) = n.\Gamma(n) = n.(n-1)! = n!. \tag{1.3}$$

Remark. *Note that* $\Gamma(n) > 0$ *always,* $\Gamma(0) = \infty$*, and for* $n \in \mathbb{N}$*,* $\Gamma(-n) = 0$*.*

(ii) **Relation between Beta and Gamma Functions.** (see [42], p. 372)

$$B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}, \quad \text{where } m, n > 0. \tag{1.4}$$

(iii) $\Gamma(0.5) = \sqrt{\pi}$. To prove this we substitute $m = n = \frac{1}{2}$ in the relation of (1.4), we have

$$B(0.5, 0.5) = \frac{\Gamma(0.5)\Gamma(0.5)}{\Gamma(1)} = (\Gamma(0.5))^2$$

and it is easy to prove that, after changing for polar, $B(0.5, 0.5) = \pi$ and consequently, $\Gamma(0.5) = \sqrt{\pi}$

1.2.4 Mittag-Leffler Function

Besides the gamma function, Euler has brought to light an additional important function, the exponential:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

According to the equation of (1.3) made in the previous section we may replace the factorial by the gamma function:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n+1)}.$$

Without difficulty this definition may be extended, where one option is given by:

$$E_\alpha(x) = \sum_{n=0}^{\infty} \frac{x^{\alpha n}}{\Gamma(n\alpha + 1)}, \qquad \alpha > 0. \tag{1.5}$$

This was introduced in the year 1903 by Mittag-Leffler [32] and consequently it is called Mittag-Leffler function. The formula $E_\alpha(x)$ in (1.5) is the one-parameter generalization of the exponential e^x. The two-parameter function of the Mittag-Leffler type, which plays a great role in the fractional calculus, was in fact introduced by Agarwal [1], and is defined by the series expansion of the form

$$E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^{\beta n}}{\Gamma(n\alpha + \beta)}, \qquad \alpha, \beta > 0. \tag{1.6}$$

It follows from this definition that there are some relationships (given e.g. in [15, 44]):

$$E_{1,1}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n+1)} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x,$$

$$E_{1,2}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n+2)} = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} = \frac{1}{x}\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} = \frac{e^x - 1}{x},$$

$$E_{1,3}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n+3)} = \sum_{n=0}^{\infty} \frac{x^n}{(n+2)!} = \frac{1}{x^2}\sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} = \frac{e^x - 1 - x}{x^2},$$

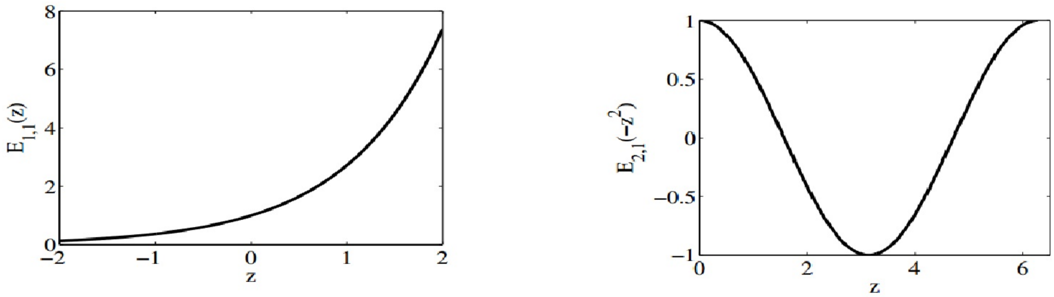
and in general

$$E_{1,m}(x) = \frac{1}{x^{m-1}}\left[e^x - \sum_{n=0}^{m-2} \frac{x^n}{n!}\right].$$

For $\beta = 1$, we obtain the Mittag-Leffler function in one parameter:

$$E_{\alpha,1}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n\alpha + 1)} = E_\alpha(x).$$

A particular cases of the Mittag-Leffler function (1.6) are the hyperbolic sine and cosine and



(a) $E_{1,1}(z)$, where $-2 < z < 2$

(b) $E_{2,1}(-z^2)$, where $0 < z < 2\pi$

Figure 1.2: Mittag-Leffler function (1.6) for various parameters

these are given by:

$$E_{1,2}(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{\Gamma(2n+1)} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \cosh(x),$$
$$E_{2,2}(x^2) = \sum_{n=0}^{\infty} \frac{x^{2n}}{\Gamma(2n+2)} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!} = \frac{1}{x}\sum_{n=0}^{\infty} \frac{x^{2n}}{2n!} = \frac{\sinh(x)}{x}.$$

Moreover, the Mittag-Leffler is related to the *error function*:

$$E_{\frac{1}{2}}(x^{\frac{1}{2}}) = e^x \left(1 + erf(x^{\frac{1}{2}})\right)$$

where the error function $erf(x)$ is given by

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

In Fig. 1.2a and Fig. 1.2b the well-known functions e^z and $\cos(z)$ are plotted and it is created for the evaluation of the Mittag-Leffler function.

1.2.4.1 Basic Properties of Mittag-Leffler Function

As a consequence of the definitions (1.5) and (1.6) the following results hold [26]

(i)

$$E_{\alpha,\beta}(x) = \beta E_{\alpha,\beta+1}(x) + \alpha x \frac{d}{dx} E_{\alpha,\beta+1}(x).$$

(ii)

$$E_{\alpha,\beta}(x) = xE_{\alpha,\alpha+\beta}(x) + \frac{1}{\Gamma(\beta)}.$$

(iii)

$$\left(\frac{d}{dx}\right)^m \left[x^{\beta-1}E_{\alpha,\beta}(x^\alpha)\right] = x^{\beta-m-1}E_{\alpha,\beta-m}(x^\alpha),$$

$$Re(\beta - m) > 0,\ m = 0, 1....$$

1.3 The Fractional Derivatives and Integrals

In this section, we will present alternative concepts to introduce a fractional derivative definition, e.g. a series expansion in terms of the standard derivative. Furthermore, several approaches to the generalization of the notion of differentiation and integration are considered.

1.3.1 The Riemann-Liouville Fractional Differintegral Operator

The Riemann-Liouville approach is based on the Cauchy formula (1.7) (see [37], p. 38 and [5], p. 64) for the n^{th} integral which uses only a simple integration so it provides a good basis for generalization.

$$I_a^n f(x) = \int_a^x \int_a^{\xi_{n-1}} \cdots \int_a^{\xi_1} f(\xi)d\xi d\xi_1 \cdots d\xi_{n-1} = \frac{1}{(n-1)!}\int_a^x (x-\xi)^{n-1} f(\xi)d\xi. \qquad (1.7)$$

Now, it is easy to get an integral of arbitrary order. The Cauchy formula (1.7), as follows: the integer n is substituted by a positive real number α and the gamma function is used instead of the factorial, a formula for fractional integration is obtained. Finally we obtain the following definitions:

Definition 1.1. Suppose that $\alpha > 0, x > a, \alpha, a, x \in \mathbb{R}$. Then the Riemann-Liouville fractional integral of order $\alpha > 0$ is defined by the following fractional operator [24, 44, 48]:

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\xi)^{\alpha-1} f(\xi) d\xi,$$
$$I_a^0 f(x) = f(x), \qquad \text{(for } \alpha = 0).$$

Definition 1.2. Accordingly, the Riemann-Liouville fractional derivative of order α is defined, for $\alpha > 0, x > a, \alpha, a, x \in \mathbb{R}$, by [24, 44, 48]:

$$D_a^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-\xi)^{n-\alpha-1} \, f(\xi) d\xi, \quad n-1 < \alpha < n \in \mathbb{N}$$

Remark. *I^α and D^α stand for I_0^α and D_0^α, respectively.*

One of the important properties associated with it is that the Riemann Liouville fractional derivative is an inverse of the integral of the same order.

The fractional integral of a power function has the following form (see [44], p. 125):

$$I_a^\alpha (x-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} (x-a)^{\beta+\alpha}, \qquad \text{for } \alpha \geq 0, \beta > -1.$$

Now, the fractional derivative of a power function has the following form:

$$D_a^\alpha (x-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (x-a)^{\beta-\alpha}. \tag{1.8}$$

So the fractional derivative of a constant takes the form

$$D_a^\alpha C = C \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}, \qquad 0 < \alpha < 1.$$

Similarly, in case of exponential function, $e^{\lambda x}$, it can be evaluated as

$$D^\alpha e^{\lambda x} = x^{-\alpha} E_{1,1-\alpha}(\lambda).$$

1.3.2 The Caputo Fractional Differential Operator

The definition of derivative provided by Riemann-Liouville has certain limitations when it is used for modeling of real-world phenomena associated with fractional differential equations. Therefore, to overcome such problems, *Caputo* proposed the following definitions:

Definition 1.3. Suppose that $\alpha > 0, x > a$, $\alpha, a, x \in \mathbb{R}$. Then the Caputo fractional derivative of order $\alpha > 0$ is defined by the following fractional operator [26, 44]

$$ {}_cD_a^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-\xi)^{n-\alpha-1} \frac{d^n}{d\xi^n} f(\xi) d\xi, & n-1 < \alpha < n \in \mathbb{N}; \\ \frac{d^n}{dx^n} f(x), & \alpha = n \in \mathbb{N}. \end{cases} $$

It is clearly from the Definition 1.3 the Caputo fractional derivative of a constant is zero. In section 1.3.1, we have shown, in Equation (1.8), the fractional derivative of a power function in the sense of Riemann-Liouville, and here the fractional derivative of a power function for the Caputo fractional have the similar form of (1.8) as:

$$ {}_cD_a^\alpha (x-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (x-a)^{\beta-\alpha}, \qquad \beta \neq 0(1)\lfloor\alpha\rfloor; $$

where $\lfloor\alpha\rfloor$ is the integer part of α.

1.3.2.1 Fractional MacLaurin Power Series Expansion for the Caputo Fractional Derivative

In this section, some kinds of fractional Taylor series expansion are defined for a general function in terms of the Caputo fractional derivative.

In order to obtain MacLaurin power series expansion (see [52], p. 3), we need the following definition:

Definition 1.4. Let $\alpha \in \mathbb{R}^+$, $\Omega \subset \mathbb{R}$ an interval such that $a \in \Omega$, $a \leq x$, $\forall x \in \Omega$. Then the following set of functions are defined (see [52], p. 4):

$$ {}_a\mathcal{I}_\alpha = \{f \in C(\Omega) : \ I_a^\alpha f(x) \, \text{exist and is finite in} \, \Omega\}, $$

and

$$ {}_a\mathcal{D}_\alpha = \{f \in C(\Omega) : {}_cD_a^\alpha f(x) \, \text{exist and is finite in} \, \Omega\}, $$

where I_a^α and ${}_cD_a^\alpha$ are, respectively, defined in Definitions 1.1 and 1.3.

Based on this observation a theorem of a formal fractional Taylor series expansion can be made.

Theorem 1.1. *Let $\alpha \in (0, 1]$, $p \in \mathbb{N}$ and $f(x)$ a continuous function in $[a, b]$ satisfying the following conditions (see [52], p. 6):*

(i) ${}_cD_a^{j\alpha} f \in C([a, b])$ and ${}_cD_a^{j\alpha} f \in {}_a\mathcal{I}_\alpha([a, b])$, $\forall j = 1(1)p$.

(ii) ${}_cD_a^{(p+1)\alpha} f(x)$ is continuous on $[a, b]$.

Then for each $x \in [a, b]$,

$$ f(x) = \sum_{j=0}^{p} {}_cD_a^{j\alpha} f(a) \frac{(x-a)^{j\alpha}}{\Gamma(j\alpha+1)} + R_p(x, a), $$

with

$$ R_p(x, a) = {}_cD_a^{(p+1)\alpha} f(\xi) \frac{(x-a)^{(p+1)\alpha}}{\Gamma((p+1)\alpha+1)}, \qquad a \leq \xi \leq x. $$

Remark. *In the above theorem, the Caputo fractional derivative ${}_cD_a^{j\alpha}$ is not equivalent to the derivative of order $j\alpha$, that is,*

$$ D^{j\cdot\alpha} f = \underbrace{D^\alpha \cdot D^\alpha \cdots D^\alpha}_{j-times}. $$

In 2007, Odibat and Shawagfeh [39] have been represented a new generalized Taylor's formula which as follows:

$$ f(x) = \sum_{m=0}^{p} {}_cD_a^{m\alpha} f(x_0) \frac{(x-x_0)^{m\alpha}}{\Gamma(m\alpha+1)} + R_p^\alpha(x), \quad 0 < \alpha \leq 1, x_0 < x \leq b $$

with

$$ R_p^\alpha(x) = {}_cD_a^{(p+1)\alpha} f(\xi) \frac{(x-x_0)^{(p+1)\alpha}}{\Gamma((p+1)\alpha+1)}, \qquad a \leq \xi \leq x. $$

Remark. *Here, the Caputo fractional derivative ${}_cD_a^{m\alpha}$ is equivalent to the derivative of order $m\alpha$.*

Theorem 1.2. *Let $f(x)$ be a function defined on the right neighborhood of a, and be an infinitely fractionally-differentiable function at a, that is to say, all ${}_cD_a^{j\alpha}f(x)$ $(j = 0, 1, 2, ...)$ exist, and are not singular at a. The formal fractional right-RL Taylor series of a function is (see [28], p. 11):*

$$f(x) = \sum_{j=0}^{\infty} {}_cD_a^{j\alpha} f(a) \frac{(x-a)^{j\alpha}}{\Gamma(j\alpha+1)}.$$

We now give some examples of fractional Taylor series.

Example 1.1 Let $f(x;\alpha) = e^{(x-a)^\alpha}$, then

$$f(x;\alpha) = \sum_{m=0}^{\infty} \frac{1}{m!}(x-a)^{m\alpha}.$$

In particular,

$$f(x;0.5) = 1 + \frac{1}{2}(x-a)^{0.5} + \frac{1}{6}(x-a) + \frac{1}{24}(x-a)^{1.5} + \frac{1}{120}(x-a)^2 + ...$$

Example 1.2 Let $f(x;\alpha) = \cos((x-a)^\alpha)$, and $g(x,\alpha) = \sin((x-a)^\alpha)$ then

$$f(x;\alpha) = 1 - \frac{(x-a)^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{(x-a)^{4\alpha}}{\Gamma(4\alpha+1)} - \frac{(x-a)^{6\alpha}}{\Gamma(6\alpha+1)} + \cdots,$$

and

$$g(x,\alpha) = \frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)} - \frac{(x-a)^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{(x-a)^{5\alpha}}{\Gamma(5\alpha+1)} - \cdots.$$

In particular,

$$f(x;0.5) = 1 - (x-a) + \frac{(x-a)^2}{2} - \frac{(x-a)^3}{6} + \frac{(x-a)^4}{24} - \frac{(x-a)^5}{120} + \cdots,$$

and

$$g(x,0.5) = \frac{2(x-a)^{1/2}}{\sqrt{\pi}} - \frac{4(x-a)^{3/2}}{3\sqrt{\pi}} + \frac{8(x-a)^{5/2}}{15\sqrt{\pi}} - \frac{16(x-a)^{7/2}}{105\sqrt{\pi}} + \frac{32(x-a)^{7/2}}{9455\sqrt{\pi}} - \cdots$$

Example 1.3 The Mittag-Leffler function $h(x;\alpha) = E_\alpha((x-a)^\alpha)$, which satisfies

$${}_cD_a^\alpha\, E_\alpha((x-a)^\alpha) = E_\alpha((x-a)^\alpha), \qquad E_\alpha(0) = 1,$$

and hence

$$h(x;\alpha) = E_\alpha((x-a)^\alpha) = \sum_{m=0}^{\infty} \frac{1}{\Gamma(m\alpha+1)}(x-a)^{m\alpha}.$$

1.3.3 The Grünwald-Letnikov Derivative

In Mathematics, the Grünwald–Letnikov derivative is a basic extension of the derivative in fractional calculus that allows one to take the derivative a non-integer number of times. It was introduced by Anton Karl Grünwald (1838–1920) from Prague, in 1867, and by Aleksey Vasilievich Letnikov (1837–1888) in Moscow in 1868. The Grunwald–Letnikov differintegral is a direct generalization of the definition of a derivative. It is more difficult to use than the Riemann–Liouville differintegral, but can sometimes be used to solve problems that the Riemann–Liouville cannot. So the Grünwald–Letnikov derivative may be succinctly written as [29]:

$${}^{G}D^\alpha y(x) = \lim_{n\to\infty} \frac{1}{h^\alpha} \sum_{k=0}^{n} g_{\alpha,k}\, y(x-kh),$$

where $g_{\alpha,k}$ are the Grünwald weights and are given as:

$$g_{\alpha,k} = \binom{\alpha}{k} = \frac{(-1)^k\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)}$$

1.4 Interpolation by Polynomial Spline Functions

Definition 1.5. Let $\mho$ be the partition $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ (for $n \geq m + 1$) of the interval $[a, b]$. Then a spline s of degree m with knots at π is a function possessing the following two properties

(a) In each subinterval $[x_i, x_{i+1})$ of $[a, b]$, s is a polynomial of degree m or less.

(b) s and its derivatives of order $1, 2, ..., m - 1$ are continuous on $[a, b]$, i.e. $s \in C^{m-1}[a, b]$.

The space all such functions is denoted by $S_m(\mho)$.

Thus, a spline function is a series of polynomial arcs of degree m or less, joined together in such a way that the function and its derivatives of orders $m - 1$ or less are continuous everywhere. The spline is, in general, a different polynomial in each of the subintervals $[x_i, x_{i+1})$ and the continuity constraint $s \in C^{m-1}[a, b]$ imposes maximal continuity on this piecewise defined function.

1.4.1 Lacunary Interpolation

Lacunary interpolation was initiated in 1957 [6]. Several researchers have studied the use of splines to solve such interpolation problems [17, 20, 21, 50]. All of these methods are global and require the solution of a large system of equations. The most appropriate method solving lacunary interpolation problems using piecewise polynomials with certain continuity properties.

As we have mentioned before, spline functions are a good tool for the numerical approximation of functions on the one hand and they also suggest new, challenging and rewarding problems on the other. Piecewise linear functions, as well as step functions, have been an important theoretical and practical tools for approximation of such functions. Lacunary interpolation by spline appears whenever observation gives scattered or irregular information about a function and it's derivatives. Also, the data in the problem of lacunary interpolation are values of the functions and of it's derivatives but without hermite condition in which consecutive derivatives are used at each nodes.

1.4.2 Modulus of Continuity

To assess the goodness of fit when we interpolate a function with a first-degree spline, it is useful to have something called the modulus of continuity of a function f. Suppose f is defined on an interval $[a, b]$. The **modulus of continuity** of f is [56]:

$$\omega(f; h) = \sup_{|x_1 - x_2| \leq h} |f(x_1) - f(x_2)|, \qquad \text{for } a \leq x_1 \leq x_2 \leq b.$$

The quantity $\omega(f; h)$ measures how much f can change over a small interval of width h. If f is continuous on $[a, b]$, then it is uniformly continuous, and $\omega(f; h)$ will tend to zero as h tends to zero. If f is not continuous, $\omega(f; h)$ will not tend to zero [56].

1.4.3 Some Theorems of Error Bounds

Let $\Delta:\ 0 = x_0 < x_1 < \cdots x_{n-1} < x_n = 1$ be the uniform partition of the interval $I = [a, b]$ with $x_{i+1} - x_i = h_i\ i = 0(1)n$. We define the class of spline functions $S_{n,k}^{(m)}$ as follows. Any element $S_\Delta \in S_{n,k}^{(m)}$ if the following conditions are fulfilled [17, 20, 21, 49, 50]:

(i) $S_\Delta \in C^m(I)$,

(ii) In each subinterval $[x_i, x_{i+1}]$, $i = 0(1)n$, $S_\Delta \in \Pi_k$, where Π_k denotes the set of polynomials of degree at most k.

Now, we will state the following theorems:

Theorem 1.3. *[49] Let $f \in C^6(I)$ and $S_\Delta \in S_{n,6}^{(3)}$ be the solution of the following problem:*

$$S_\Delta(x_i) = y_i, \quad S_\Delta^{(q)}(x_i) = y_i^{(q)}, \quad q = 2, 3;\ i = 0(1)n,$$
$$S'_\Delta(x_0) = y'_0, \quad S'_\Delta(x_n) = y'_n.$$

Then

$$\left|S_\Delta^{(q)}(x) - f^{(q)}(x)\right| \leq k_1 h^{5-q} \omega_6(h), \quad q = 0(1)5,$$

where

$$k_1 = \begin{cases} 604, & when \quad x \in [x_0, x_1] \\ 24, & when \quad x \in [x_i, x_{i+1}],\ i = 1(1)n-2 \\ 35h, & when \quad x \in [x_{n-1}, x_n] \end{cases}$$

and $\omega_6(\cdot)$ is the modulus of continuity of $f^{(6)}$.

Theorem 1.4. *[50] Let $S_\Delta \in S^{(2)}_{n,6}$ be the solution of the following problem:*

$$S^{(q)}_\Delta(x_i) = y^{(q)}_i, \quad q = 0, 1, 2, 4;\ i = 0(1)n.$$

Then for $f \in C^6(I)$, we have

$$\left|S^{(q)}_i - f^{(q)}\right| \leq c_{i,q} h^{6-q} \omega_6(h), \quad q = 0(1)6,\ i = 0(1)n,$$

where $\omega_6(\cdot)$ is the modulus of continuity of $f^{(6)}$, and the constants $c_{i,q}$ are given in the following table:

	$c_{i,0}$	$c_{i,1}$	$c_{i,2}$	$c_{i,3}$	$c_{i,4}$	$c_{i,5}$	$c_{i,6}$
$0 \leq i \leq n-2$	$\frac{49}{720}$	$\frac{1}{3}$	$\frac{11}{8}$	$\frac{55}{12}$	$\frac{23}{2}$	19	15
$i = n-1$	$\frac{245}{1008}$	$\frac{53}{40}$	$\frac{739}{120}$	$\frac{118}{5}$	71	$\frac{311}{2}$	218

1.4.4 Advantages of Spline Functions

Spline solution has its own advantages. For example, once the solution has been computed, the information required for spline interpolation between mesh points is available. This is particularly important when the solution of the BVP is required at different locations in the interval $[a, b]$. This approach has the added advantage that it not only provides continuous approximations to $y(x)$, but also to y' and higher derivatives at every point of the range of integration. Also, the C^∞-differentiability of the trigonometric part of non-polynomial splines compensates for the loss of smoothness inherited by polynomial splines. Moreover, we may say:

(i) A graph of the constructed function passes through every point of the given array.

(ii) The constructed function is uniquely determined by the given array.

(iii) A degree of the polynomials used for description of the interpolating function is independent of the knot's number, consequently, does not change as the number increases.

Chapter Two

Numerical Solution of Fractional Differential Equations by using Fractional Lacunary Spline Functions

Chapter 2

Numerical Solution of FDEs by using Fractional Lacunary Spline Functions

2.1 Introduction

Fractional differential equations are gaining considerable importance due to their wide range of applications in the fields of physics, engineering [5, 24], chemistry, and/or biochemistry [57], optimal control [53–55], medicine [19] and biology [44]. Several numerical techniques such as Adomian decomposition method (ADM) [27, 33], Adams-Bashforth-Moulton method [34, 36], fractional difference method [37], fractional spline function of a polynomial form [31, 58], and variational iteration method [15, 44] have been developed for solving non-linear functional equations in general and solving fractional differential equations in particular.

In view of successful application of spline functions of polynomial form in system analysis [31], fractional differential equations [25, 58], and delay differential equations of fractional order [46], we notice that it should be applicable to solve fractional differential equations with the idea of the lacunary interpolation. For details about lacunary interpolation, we may refer to ([17, 20, 21, 49, 50]).

In this chapter, we investigate numerical solution of fractional differential equations (FDEs) using the idea of lacunary interpolation.

2.2 Generalized quartic fractional spline interpolation with applications

As we have mentioned in the above section, there are several methods for solving FDEs. So, we introduce the following method:

2.2.1 Descriptions

Given the mesh points, $\triangle : 0 = x_0 < x_1 < \cdots < x_n = 1$ with $x_{k+1} - x_k = h$, $k = 0(1)n-1$, and real numbers $\left\{y_k, D^{\alpha}y_k, D^{4\alpha}y_k\right\}_{k=0}^{n}$ associated with the knots, where $y_k = y(x_k)$. We are going to construct spline interpolant $S_{\triangle}$ for which $D^{m\alpha}S_{\triangle}(x_i) = D^{m\alpha}y_i$, $i = 0(1)n$, and $m = 0, 1, 4$. This construction is given in the following two cases:

Case 1

In this case, we suppose that the conditions of Theorem 1.1 are satisfied with $p = 4$, and then we can define the spline interpolant as follows:

$$\begin{aligned} S_{\triangle} = S_k(x) = & y_k + \frac{(x-x_k)^{\alpha}}{\Gamma(\alpha+1)} D^{\alpha} y_k + a_k \frac{(x-x_k)^{2\alpha}}{\Gamma(2\alpha+1)} \\ & + b_k \frac{(x-x_k)^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{(x-x_k)^{4\alpha}}{\Gamma(4\alpha+1)} D^{4\alpha} y_k, \end{aligned} \tag{2.1}$$

where $x_k \leq x \leq x_{k+1}$ and $k = 0(1)n-1$.

2.2.2 Existence and Uniqueness

If we require that $S_{\triangle}(x)$ and $D^{\alpha}S_{\triangle}(x)$ is continuous on $[0, 1]$, then it is easy to prove that formula (2.1) exists and is unique. That is, clear from the continuity conditions of $S_{\triangle}(x)$ and $D^{\alpha}S_{\triangle}(x)$, we get:

$$y_{k+1} = y_k + \frac{h^{\alpha}}{\Gamma(\alpha+1)} D^{\alpha} y_k + a_k \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} + b_k \frac{h^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{h^{4\alpha}}{\Gamma(4\alpha+1)} D^{4\alpha} y_k, \tag{2.2}$$

and from Equation (2.1), we have

$$D^{\alpha}y_{k+1} = D^{\alpha}y_k + \frac{h^{\alpha}}{\Gamma(\alpha+1)}a_k + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)}b_k + \frac{h^{3\alpha}}{\Gamma(3\alpha+1)}D^{4\alpha}y_k. \tag{2.3}$$

The coefficients a_k and b_k are determined in terms of the given data using the continuity conditions of $S_{\triangle}(x)$ and $D^{\alpha}S_{\triangle}(x)$. Thus we have

$$a_k = \frac{\frac{1}{\Gamma(2\alpha+1)}A_k - \frac{h^{\alpha}}{\Gamma(3\alpha+1)}B_k}{k_1 h^{2\alpha}}, \tag{2.4}$$

and

$$b_k = \frac{\frac{1}{\Gamma(2\alpha+1)}B_k - \frac{h^{-\alpha}}{\Gamma(\alpha+1)}A_k}{k_1 h^{2\alpha}}, \tag{2.5}$$

where

$$\begin{aligned} k_1 &= \frac{1}{\Gamma(2\alpha+1)\Gamma(2\alpha+1)} - \frac{1}{\Gamma(\alpha+1)\Gamma(3\alpha+1)}, \\ A_k &= y_{k+1} - y_k - \frac{h^{\alpha}}{\Gamma(\alpha+1)}D^{\alpha}y_k - \frac{h^{4\alpha}}{\Gamma(4\alpha+1)}D^{4\alpha}y_k, \end{aligned} \tag{2.6}$$

and

$$B_k = D^{\alpha}y_{k+1} - D^{\alpha}y_k - \frac{h^{3\alpha}}{\Gamma(3\alpha+1)}D^{4\alpha}y_k, \quad \text{for } k = 0(1)n-1. \tag{2.7}$$

Note 2.1. For $\alpha = \frac{1}{2}$, we have

$$\begin{aligned} S_{\triangle} = S_k(x) =& y_k + D^{1/2}y_k\frac{2(x-x_k)^{1/2}}{\sqrt{\pi}} + a_k(x-x_k) \\ &+ b_k\frac{4(x-x_k)^{3/2}}{3\sqrt{\pi}} + (D^{1/2})^4 y_k\frac{(x-x_k)^2}{2}, \end{aligned}$$

where $x_k \leq x \leq x_{k+1}$ and $k = 0(1)n-1$.

$$y_{k+1} = y_k + \frac{2}{\sqrt{\pi}} h^{1/2} D^{1/2} y_k + h a_k + \frac{4}{3\sqrt{\pi}} h^{3/2} b_k + \frac{1}{2} h^2 (D^{1/2})^4 y_k, \tag{2.8}$$

and

$$D^{1/2} y_{k+1} = D^{1/2} y_k + \frac{2}{\sqrt{\pi}} h^{1/2} a_k + h b_k + \frac{4}{3\sqrt{\pi}} h^{3/2} (D^{1/2})^4 y_k. \tag{2.9}$$

The constants a_k and b_k are given

$$a_k = \frac{\sqrt{\pi}\left(3\sqrt{\pi} A_k - 4 B_k h^{1/2}\right)}{h(3\pi - 8)},$$

and

$$b_k = \frac{3\sqrt{\pi}\left(\sqrt{\pi} B_k - 2 A_k h^{-1/2}\right)}{h(3\pi - 8)},$$

where

$$A_k = h a_k + \frac{4}{3\sqrt{\pi}} h^{3/2} b_k,$$

and

$$B_k = \frac{2}{\sqrt{\pi}} h^{1/2} a_k + h b_k, \quad \text{for } k = 0(1)n-1.$$

2.2.3 Error Bounds

Suppose that the conditions of Theorem 1.1 are satisfied with $p = 4$ and $D^{m\alpha} S_{\triangle}(x_i) = D^{m\alpha} y_i$, $\alpha \in (0,1]$, $m = 0, 1, 4;\ i = 0(1)n-1$. We shall prove the following:

Theorem 2.1. *Let $S_k(x)$ be the fractional spline interpolant of the polynomial form (2.1) solving the lacunary case $(0, \alpha, 4\alpha)$. Then for all $x \in [0,1]$ the inequality*

$$|D^{m\alpha} S_{\triangle}(x) - D^{m\alpha} y(x)| \leq c_{m\alpha} h^{(4-m)\alpha}\, \omega_{4\alpha}(h),$$

holds for all $m = 0(1)4$, and $\alpha \in (0, 1]$, where $\omega_{4\alpha}(h)$ is the modulus of continuity of $D^{4\alpha}y(x)$, and

$$c_0 = \frac{k_2}{k_1\Gamma(2\alpha+1)} + \frac{k_3}{k_1\Gamma(3\alpha+1)} + \frac{1}{\Gamma(4\alpha+1)}, \quad c_\alpha = \frac{k_2}{k_1\Gamma(\alpha+1)} + \frac{k_3}{k_1\Gamma(2\alpha+1)} + \frac{1}{\Gamma(3\alpha+1)},$$
$$c_{2\alpha} = \frac{k_2}{k_1} + \frac{k_3}{k_1\Gamma(\alpha+1)} + \frac{1}{\Gamma(2\alpha+1)}, \quad c_{3\alpha} = \frac{k_3}{k_1} + \frac{1}{\Gamma(\alpha+1)}, \quad c_{4\alpha} = 1.$$

Note 2.2. For $\alpha = \frac{1}{2}$, we have

$$|\left(D^{1/2}\right)^m S_\triangle(x) - \left(D^{1/2}\right)^m y(x)| \le c_{\frac{m}{2}} h^{2-\frac{m}{2}}\, \omega(h),$$

where $m = 0(1)4$ and $\omega(h)$ is the modulus of continuity of $(D^{1/2})^4 y(x)$, and

$$c_0 = \frac{9\pi + 48\sqrt{\pi} + 4}{9\pi - 24}, \quad c_{1/2} = \frac{14\sqrt{\pi}}{3\pi - 8}, \quad c_1 = \frac{27\sqrt{\pi} + 68}{18\pi - 48}, \quad c_{3/2} = \frac{7\sqrt{\pi}}{3\pi - 8} + \frac{2}{\sqrt{\pi}}, \quad c_2 = 1.$$

To prove Theorem 2.1, we shall need the following lemma,

Lemma 2.2. *The following estimates are valid:*

$$\left|a_k - D^{2\alpha}y_k\right| \le \frac{k_2}{k_1} h^{2\alpha}\ \omega_{4\alpha}(h), \tag{2.10}$$
$$\left|b_k - D^{3\alpha}y_k\right| \le \frac{k_3}{k_1} h^{\alpha}\ \omega_{4\alpha}(h), \tag{2.11}$$

for $k = 0(1)n-1$, where

$$k_2 = \left(\frac{1}{\Gamma(2\alpha+1)\Gamma(4\alpha+1)} + \frac{1}{\Gamma(3\alpha+1)\Gamma(3\alpha+1)}\right),$$

and

$$k_3 = \left(\frac{1}{\Gamma(2\alpha+1)\Gamma(3\alpha+1)} + \frac{1}{\Gamma(\alpha+1)\Gamma(4\alpha+1)}\right).$$

Proof. From (2.6) we can find

$$
\begin{aligned}
\left|a_k - D^{2\alpha} y_k\right| = &\left|\frac{1}{k_1 h^{2\alpha}}\left[\frac{1}{\Gamma(2\alpha+1)}\left(y_{k+1} - y_k - \frac{h^{\alpha}}{\Gamma(\alpha+1)} D^{\alpha} y_k - \frac{h^{4\alpha}}{\Gamma(4\alpha+1)} D^{4\alpha} y_k\right)\right.\right.\\
&\left.\left. - \frac{h^{\alpha}}{\Gamma(3\alpha+1)}\left(D^{\alpha} y_{k+1} - D^{\alpha} y_k - \frac{h^{3\alpha}}{\Gamma(3\alpha+1)} D^{4\alpha} y_k\right)\right] - D^{2\alpha} y_k\right|\\
= &\left|\frac{1}{k_1 h^{2\alpha}}\left[\frac{1}{\Gamma(2\alpha+1)}\left(y_{k+1} - y_k - \frac{h^{\alpha}}{\Gamma(\alpha+1)} D^{\alpha} y_k - \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} D^{2\alpha} y_k\right.\right.\right.\\
&\left. - \frac{h^{4\alpha}}{\Gamma(4\alpha+1)} D^{4\alpha} y_k\right) - \frac{h^{\alpha}}{\Gamma(3\alpha+1)}\left(D^{\alpha} y_{k+1} - D^{\alpha} y_k\right.\\
&\left.\left.\left. - \frac{h^{\alpha}}{\Gamma(\alpha+1)} D^{2\alpha} y_k - \frac{h^{3\alpha}}{\Gamma(3\alpha+1)} D^{4\alpha} y_k\right)\right]\right|.
\end{aligned}
\tag{2.12}
$$

Taking:

$$
y_{k+1} = y_k + \frac{h^{\alpha}}{\Gamma(\alpha+1)} D^{\alpha} y_k + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} D^{2\alpha} y_k + \frac{h^{3\alpha}}{\Gamma(3\alpha+1)} D^{3\alpha} y_k + \frac{h^{4\alpha}}{\Gamma(4\alpha+1)} D^{4\alpha} y(\xi_k),
$$

and

$$
D^{\alpha} y_{k+1} = D^{\alpha} y_k + \frac{h^{\alpha}}{\Gamma(\alpha+1)} D^{2\alpha} y_k + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} D^{3\alpha} y_k + \frac{h^{3\alpha}}{\Gamma(3\alpha+1)} D^{4\alpha} y(\eta_k),
$$

where $x_k < \xi_k, \eta_k < x_{k+1}$. Then (2.12) becomes

$$
\begin{aligned}
\left|a_k - D^{2\alpha} y_k\right| &\le \frac{1}{k_1 h^{2\alpha}}\left[\frac{h^{4\alpha}}{\Gamma(2\alpha+1)\Gamma(4\alpha+1)}\left|D^{4\alpha} y(\xi_k) - D^{4\alpha} y_k\right|\right.\\
&\quad\left. + \frac{h^{4\alpha}}{\Gamma(3\alpha+1)\Gamma(3\alpha+1)}\left|D^{4\alpha} y(\eta_k) - D^{4\alpha} y_k\right|\right]\\
&\le \frac{h^{2\alpha}}{k_1}\left[\frac{1}{\Gamma(2\alpha+1)\Gamma(4\alpha+1)} + \frac{1}{\Gamma(3\alpha+1)\Gamma(3\alpha+1)}\right]\omega_{4\alpha}(h)\\
&= \frac{k_2}{k_1}\, h^{2\alpha}\, \omega_{4\alpha}(h).
\end{aligned}
$$

Similarly, after using (2.7), we can easily prove the second part of the lemma. Thus, we have proved the lemma. □

Note 2.3. For $\alpha = \frac{1}{2}$, we have

$$
\begin{aligned}
\left|a_k - (D^{1/2})^2 y_k\right| &\leq \frac{9\pi + 32}{18\pi - 48}\, h^{3/2}\omega(h),\\
\left|b_k - (D^{1/2})^3 y_k\right| &\leq \frac{7\sqrt{\pi}}{3\pi - 8}\, h\,\omega(h),
\end{aligned}
$$

for $k = 0(1)n-1$.

Proof of theorem 2.1. In view of the above lemma, we can see that, for $x_k \leq x \leq x_{k+1}$ and $k = 0(1)n-1$,

$$
\begin{aligned}
|S_k(x) - y(x)| = &\left| y_k + \frac{(x-x_k)^{\alpha}}{\Gamma(\alpha+1)} D^{\alpha} y_k + a_k \frac{(x-x_k)^{2\alpha}}{\Gamma(2\alpha+1)} + b_k \frac{(x-x_k)^{3\alpha}}{\Gamma(3\alpha+1)} \right.\\
&+ \frac{(x-x_k)^{4\alpha}}{\Gamma(4\alpha+1)} D^{4\alpha} y_k - y_k - \frac{(x-x_k)^{\alpha}}{\Gamma(\alpha+1)} D^{\alpha} y_k - D^{2\alpha} y_k \frac{(x-x_k)^{2\alpha}}{\Gamma(2\alpha+1)}\\
&\left. - D^{3\alpha} y_k \frac{(x-x_k)^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{(x-x_k)^{4\alpha}}{\Gamma(4\alpha+1)} D^{4\alpha} y(\xi_k) \right|\\
&\leq \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} \left|a_k - D^{2\alpha} y_k\right| + \frac{h^{3\alpha}}{\Gamma(3\alpha+1)} \left|b_k - D^{3\alpha} y_k\right| + \frac{h^{4\alpha}}{\Gamma(4\alpha+1)} \omega_{4\alpha}(h)\\
&\leq \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} \cdot \frac{k_2}{k_1} h^{2\alpha}\ \omega_{4\alpha}(h) + \frac{h^{3\alpha}}{\Gamma(3\alpha+1)} \cdot \frac{k_3}{k_1} h^{\alpha}\ \omega_{4\alpha}(h) + \frac{h^{4\alpha}}{\Gamma(4\alpha+1)} \omega_{4\alpha}(h).
\end{aligned}
$$

By using (2.10) and (2.11), the last equation leads to

$$
|S_k(x) - y(x)| \leq \left(\frac{k_2}{k_1 \Gamma(2\alpha+1)} + \frac{k_3}{k_1 \Gamma(3\alpha+1)} + \frac{1}{\Gamma(4\alpha+1)} \right) h^{4\alpha}\, \omega_{4\alpha}(h), \tag{2.13}
$$

and

$$
\begin{aligned}
|D^{\alpha} S_k(x) - D^{\alpha} y(x)| = &\left| D^{\alpha} y_k + \frac{(x-x_k)^{\alpha}}{\Gamma(\alpha+1)} a_k + \frac{(x-x_k)^{2\alpha}}{\Gamma(2\alpha+1)} b_k + \frac{(x-x_k)^{3\alpha}}{\Gamma(3\alpha+1)} D^{4\alpha} y_k \right.\\
&\left. - D^{\alpha} y_k - \frac{(x-x_k)^{\alpha}}{\Gamma(\alpha+1)} D^{2\alpha} y_k - \frac{(x-x_k)^{2\alpha}}{\Gamma(2\alpha+1)} D^{3\alpha} y_k - \frac{(x-x_k)^{3\alpha}}{\Gamma(3\alpha+1)} D^{4\alpha} y(\xi_k) \right|\\
&\leq \frac{h^{\alpha}}{\Gamma(\alpha+1)} \left|a_k - D^{2\alpha} y_k\right| + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} \left|b_k - D^{3\alpha} y_k\right| + \frac{h^{3\alpha}}{\Gamma(3\alpha+1)} \omega_{4\alpha}(h)\\
&\leq \left(\frac{k_2}{k_1 \Gamma(\alpha+1)} + \frac{k_3}{k_1 \Gamma(2\alpha+1)} + \frac{1}{\Gamma(3\alpha+1)} \right) h^{3\alpha}\, \omega_{4\alpha}(h).
\end{aligned} \tag{2.14}
$$

Similarly,

$$\begin{aligned}\left|D^{2\alpha}S_k(x)-D^{2\alpha}y(x)\right| &\le \left|a_k-D^{2\alpha}y_k\right|+\frac{h^{\alpha}}{\Gamma(\alpha+1)}\left|b_k-D^{3\alpha}y_k\right|+\frac{h^{2\alpha}}{\Gamma(2\alpha+1)}\omega_{4\alpha}(h)\\ &\le\left(\frac{k_2}{k_1}+\frac{k_3}{k_1\Gamma(\alpha+1)}+\frac{1}{\Gamma(2\alpha+1)}\right)h^{2\alpha}\,\omega_{4\alpha}(h), \qquad (2.15)\end{aligned}$$

$$\begin{aligned}\left|D^{3\alpha}S_k(x)-D^{3\alpha}y(x)\right| &\le \left|b_k-D^{3\alpha}y_k\right|+\frac{h^{\alpha}}{\Gamma(\alpha+1)}\omega_{4\alpha}(h)\\ &\le\left(\frac{k_3}{k_1}+\frac{1}{\Gamma(\alpha+1)}\right)h^{\alpha}\,\omega_{4\alpha}(h), \qquad (2.16)\end{aligned}$$

and finally,

$$\left|D^{4\alpha}S_k(x)-D^{4\alpha}y(x)\right|\le\omega_{4\alpha}(h). \qquad (2.17)$$

This completes the proof. □

Case 2

In this case, we suppose that the conditions of Theorem 1.1 are fulfilled with $p=5$, then we can define the spline interpolant as follows:

$$\begin{aligned}S_{\triangle}=S_k(x)=&y_k+\frac{(x-x_k)^{\alpha}}{\Gamma(\alpha+1)}D^{\alpha}y_k+a_k\frac{(x-x_k)^{2\alpha}}{\Gamma(2\alpha+1)}\\ &+b_k\frac{(x-x_k)^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{(x-x_k)^{4\alpha}}{\Gamma(4\alpha+1)}D^{4\alpha}y_k+c_k\frac{(x-x_k)^{5\alpha}}{\Gamma(5\alpha+1)}, \qquad (2.18)\end{aligned}$$

where $x_k\le x\le x_{k+1}$ and $k=0(1)n-1$.

Let

$$c_k=\Gamma(\alpha+1)h^{-\alpha}\left[D^{4\alpha}y_{k+1}-D^{4\alpha}y_k\right].$$

It can be easily shown that

$$\left|c_k-D^{5\alpha}y_k\right|\le\omega_{5\alpha}(h), \qquad (2.19)$$

where $\omega_{5\alpha}(h)$ is the modulus of continuity of $D^{5\alpha}y(x)$.

Now, if $S_{\triangle}(x)\in C[0,1]$ and $S^{\alpha}_{\triangle}(x)\in C[0,1]$ then the *existence* and *uniqueness* of $S_{\triangle}(x)$ is easy to be proved, since here a_k and b_k are uniquely determined by

$$a_k = \frac{\frac{1}{\Gamma(2\alpha+1)}A_k - \frac{h^\alpha}{\Gamma(3\alpha+1)}B_k}{k_1 h^{2\alpha}}, \tag{2.20}$$

$$b_k = \frac{\frac{1}{\Gamma(2\alpha+1)}B_k - \frac{h^{-\alpha}}{\Gamma(\alpha+1)}A_k}{k_1 h^{2\alpha}}, \tag{2.21}$$

where

$$k_1 = \frac{1}{\Gamma(2\alpha+1)\Gamma(2\alpha+1)} - \frac{1}{\Gamma(\alpha+1)\Gamma(3\alpha+1)},$$

$$A_k = y_{k+1} - y_k - \frac{h^\alpha}{\Gamma(\alpha+1)}D^\alpha y_k - \frac{h^{4\alpha}}{\Gamma(4\alpha+1)}D^{4\alpha} y_k - \frac{h^{5\alpha}}{\Gamma(5\alpha+1)}c_k, \tag{2.22}$$

and $$B_k = D^\alpha y_{k+1} - D^\alpha y_k - \frac{h^{3\alpha}}{\Gamma(3\alpha+1)}D^{4\alpha} y_k - \frac{h^{4\alpha}}{\Gamma(4\alpha+1)}c_k. \tag{2.23}$$

Note 2.4. For $\alpha = \frac{1}{2}$, we have

$$\begin{aligned} S_\triangle = S_k(x) =& y_k + D^{1/2}y_k\frac{2(x-x_k)^{1/2}}{\sqrt{\pi}} + a_k(x-x_k) \\ &+ b_k\frac{4(x-x_k)^{3/2}}{3\sqrt{\pi}} + (D^{1/2})^4 y_k\frac{(x-x_k)^2}{2} + c_k\frac{8(x-x_k)^{5/2}}{15\sqrt{\pi}}, \end{aligned}$$

where $x_k \le x \le x_{k+1}$ and $k = 0(1)n-1$.

Let

$$c_k = \frac{\sqrt{\pi}}{2}h^{-1/2}\left[\left(D^{1/2}\right)^4 y_{k+1} - \left(D^{1/2}\right)^4 y_k\right].$$

It can be easily shown that

$$\left|c_k - \left(D^{1/2}\right)^5 y_k\right| \le \omega(h),$$

where $\omega(h)$ is the modulus of continuity of $\left(D^{1/2}\right)^5 y(x)$.

The constants a_k and b_k are given by

$$a_k = \frac{\sqrt{\pi}\left(3\sqrt{\pi}A_k - 4B_k h^{1/2}\right)}{h(3\pi-8)},$$

and

$$b_k = \frac{3\sqrt{\pi}\left(\sqrt{\pi}B_k - 2A_k h^{-1/2}\right)}{h(3\pi - 8)},$$

where

$$A_k = y_{k+1} - y_k - \frac{2}{\sqrt{\pi}}h^{1/2}(D^{1/2})y_k - \frac{1}{2}h^2(D^{1/2})^4 y_k - \frac{8}{15\sqrt{\pi}}h^{5/2}c_k,$$

and

$$B_k = D^{(1/2)}y_{k+1} - D^{(1/2)}y_k - \frac{4}{3\sqrt{\pi}}h^{3/2}(D^{1/2})^4 y_k - \frac{1}{2}h^2 c_k.$$

Then we have the following lemma:

Lemma 2.3. *The following estimates can be obtained*

$$\left|a_k - D^{2\alpha}y_k\right| \leq \frac{k_4}{k_1}\, h^{2\alpha}\ \omega_{4\alpha}(h), \tag{2.24}$$

$$\left|b_k - D^{3\alpha}y_k\right| \leq \frac{k_5}{k_1}\, h^{\alpha}\ \omega_{4\alpha}(h), \tag{2.25}$$

for $k = 0(1)n - 1$, *where*

$$k_4 = \left(\frac{1}{\Gamma(2\alpha + 1)\Gamma(5\alpha + 1)} + \frac{1}{\Gamma(3\alpha + 1)\Gamma(4\alpha + 1)}\right)$$

and

$$k_5 = \left(\frac{1}{\Gamma(2\alpha + 1)\Gamma(4\alpha + 1)} + \frac{1}{\Gamma(\alpha + 1)\Gamma(5\alpha + 1)}\right).$$

Proof. Use (2.19), (2.20), (2.22), and (2.23) to obtain the first inequality and use (2.19), (2.21), (2.22), and (2.23) to obtain the second inequality and follow the steps of the lemma 2.2. Thus we can prove the lemma. □

Note 2.5. For $\alpha = \frac{1}{2}$, we have

$$\left|a_k - (D^{1/2})^2 y_k\right| \leq \frac{18\sqrt{\pi}}{5(3\pi - 8)} h^{3/2}\omega(h),$$
$$\left|b_k - (D^{1/2})^3 y_k\right| \leq \frac{3}{16}\frac{15\pi + 32}{3\pi - 8} h\omega(h).$$

And then we can conclude the following theorem:

Theorem 2.4. *Let $S_k(x)$ be the fractional spline interpolant of the polynomial form (2.1) solving the lacunary case $(0, \alpha, 4\alpha)$ for which the conditions of Theorem 1.1 are satisfied with $p = 5$. Then for all $x \in [0, 1]$ the inequality*

$$|D^{m\alpha} S_{\triangle}(x) - D^{m\alpha} y(x)| \leq c_{m\alpha} h^{(5-m)\alpha}\, \omega_{5\alpha}(h),$$

holds for all $m = 0(1)5$, and $\alpha \in (0, 1]$, where $\omega_{5\alpha}(h)$ is the modulus of continuity of $D^{5\alpha} y(x)$, and

$$c_0 = \frac{k_4}{k_1\Gamma(2\alpha + 1)} + \frac{k_5}{k_1\Gamma(3\alpha + 1)} + \frac{1}{\Gamma(5\alpha + 1)}, \quad c_\alpha = \frac{k_4}{k_1\Gamma(\alpha + 1)} + \frac{k_5}{k_1\Gamma(2\alpha + 1)} + \frac{1}{\Gamma(4\alpha + 1)},$$
$$c_{2\alpha} = \frac{k_4}{k_1} + \frac{k_5}{k_1\Gamma(\alpha + 1)} + \frac{1}{\Gamma(3\alpha + 1)}, \quad c_{3\alpha} = \frac{k_5}{k_1} + \frac{1}{\Gamma(2\alpha + 1)}, \quad c_{4\alpha} = \frac{1}{\Gamma(\alpha + 1)}, \quad c_{5\alpha} = 1.$$

Proof. Proceed as in Theorem 2.1. □

Note 2.6. For $\alpha = \frac{1}{2}$, we have

$$|\,(D^\alpha)^m\, S_{\triangle}(x) - (D^\alpha)^m\, y(x)| \leq c_{m\alpha} h^{2.5 - m\alpha}\omega(h),$$

where $\omega(h)$ is the modulus of continuity of $(D^{1/2})^5 y(x)$, and

$$c_0 = \frac{75\pi + 72\sqrt{\pi} + 160}{20(3\pi - 8)} + \frac{8}{15\sqrt{\pi}}, \quad c_{1/2} = \frac{36}{5(3\pi - 8)} + \frac{3}{16}\frac{15\pi + 32}{3\pi - 8} + \frac{1}{2},$$

$$c_1 = \frac{18\sqrt{\pi}}{5(3\pi - 8)} + \frac{3}{8\sqrt{\pi}}\frac{15\pi + 32}{3\pi - 8} + \frac{4}{3\sqrt{\pi}}, \quad c_{3/2} = \frac{3}{18}\frac{15\pi + 32}{3\pi - 8} + 1, \quad c_2 = \frac{2}{\sqrt{\pi}}, \quad c_{5/2} = 1.$$

2.2.4 Numerical Illustrations

We now consider some numerical examples illustrating the solution using our fractional spline method. All calculations are implemented with MATLAB 12b.

Example 2.1 Consider the linear fractional differential equation

$$D^2 y(x) + 2\,D^{\alpha} y(x) + y(x) = 2x + \frac{4}{\Gamma(4-\alpha)} x^{3-\alpha} + \frac{1}{3} x^3, \quad 0 < \alpha \le 1, \tag{2.26}$$

subject to

$$y(0) = y'(0) = 0.$$

It is easily verified that the exact solution of this problem is

$$y(x) = \frac{1}{3} x^3.$$

The maximal absolute errors obtained for $\alpha = 0.5, 0.8$, and for $0 \le x \le 1$ in each case and these are shown in Tables 2.1–2.4, to illustrate the accuracy of the proposed method. Note that $|e^{m\alpha}(x)| = |D^{m\alpha} S_k(x) - D^{m\alpha} y(x)|$, where $m = 0(1)4$ for case 1, and $m = 0(1)5$ for case 2.

Table 2.1: Maximal absolute errors in case 1 where $\alpha = 0.5$ for Example 2.1.

h	$\lvert e(x)\rvert$	$\lvert e^{\alpha}(x)\rvert$	$\lvert e^{2\alpha}(x)\rvert$	$\lvert e^{3\alpha}(x)\rvert$	$\lvert e^{4\alpha}(x)\rvert$
0.1	$5.4910E-02$	$1.1015E-01$	$2.7105E-01$	$6.2211E-01$	$9.0972E-01$
0.01	$5.4910E-05$	$3.4832E-04$	$2.7105E-03$	$1.9673E-02$	$9.9861E-02$
0.001	$5.4910E-08$	$1.1015E-06$	$2.7105E-05$	$6.2211E-04$	$8.6722E-03$

Table 2.2: Maximal absolute errors in case 2 where $\alpha = 0.5$ for Example 2.1.

h	$\lvert e(x)\rvert$	$\lvert e^{\alpha}(x)\rvert$	$\lvert e^{2\alpha}(x)\rvert$
0.1	$4.2117E-02$	$1.1394E-01$	$3.8320E-01$
0.01	$4.2117E-05$	$3.6031E-04$	$3.8320E-03$
0.001	$1.3318E-07$	$1.1394E-06$	$3.8320E-05$
h	$\lvert e^{3\alpha}(x)\rvert$	$\lvert e^{4\alpha}(x)\rvert$	$\lvert e^{5\alpha}(x)\rvert$
0.1	$1.5609E-01$	$2.5464E-01$	$7.1364E-01$
0.01	$4.9362E-03$	$2.5464E-02$	$2.2567E-01$
0.001	$1.5609E-04$	$2.5464E-03$	$7.1364E-02$

Table 2.3: Maximal absolute errors in case 1 where $\alpha = 0.8$ for Example 2.1.

h	$\lvert e(x)\rvert$	$\lvert e^{\alpha}(x)\rvert$	$\lvert e^{2\alpha}(x)\rvert$	$\lvert e^{3\alpha}(x)\rvert$	$\lvert e^{4\alpha}(x)\rvert$
0.1	$1.1003E-04$	$2.1189E-03$	$5.9215E-02$	$1.1003E-01$	$2.3372E-01$
0.01	$7.1990E-08$	$5.7815E-07$	$2.5117E-05$	$1.9972E-03$	$9.9881E-03$
0.001	$6.9302E-12$	$2.1003E-10$	$1.8135E-07$	$5.3441E-05$	$9.9985E-04$

Table 2.4: Maximal absolute errors in case 2 where $\alpha = 0.8$ for Example 2.1.

h	$\lvert e(x)\rvert$	$\lvert e^{\alpha}(x)\rvert$	$\lvert e^{2\alpha}(x)\rvert$
0.1	$5.3811E-05$	$3.1091E-04$	$3.2220E-03$
0.01	$9.5419E-09$	$1.7701E-08$	$3.9928E-07$
0.001	$8.6319E-13$	$1.4098E-10$	$2.0171E-08$
h	$\lvert e^{3\alpha}(x)\rvert$	$\lvert e^{4\alpha}(x)\rvert$	$\lvert e^{5\alpha}(x)\rvert$
0.1	$6.1608E-03$	$2.7461E-02$	$1.1644E-01$
0.01	$3.9514E-04$	$2.1425E-03$	$8.9969E-02$
0.001	$5.8103E-07$	$2.9424E-05$	$9.9389E-03$

Example 2.2 Consider the fractional differential equation

$$D^{\alpha}y(x) = x^4 - \frac{1}{2}x^3 + \frac{24}{\Gamma(4-\alpha)}x^{3-\alpha} + \frac{3}{\Gamma(5-\alpha)}x^{4-\alpha} - y(x), \quad 0 < \alpha < 1, \tag{2.27}$$

with the initial condition $y(0) = 0$. The exact solution is

$$y(x) = x^4 - \frac{1}{2}x^3.$$

Similarly, the maximal absolute errors obtained, for case 1, case 2 and for $\alpha = 0.5$, are shown in Tables 2.5–2.8, respectively, with $|e^{m\alpha}(x)| = |D^{m\alpha}S_k(x) - D^{m\alpha}y(x)|$, where $m = 0(1)4$ for case 1, and $m = 0(1)5$ for case 2.

Table 2.5: Maximal absolute error in case 1 where $\alpha = 0.5$ for Example 2.2.

h	$\lvert e(x)\rvert$	$\lvert e^{\alpha}(x)\rvert$	$\lvert e^{2\alpha}(x)\rvert$	$\lvert e^{3\alpha}(x)\rvert$	$\lvert e^{4\alpha}(x)\rvert$
0.1	$7.4128E-03$	$1.4870E-02$	$3.6591E-01$	$8.3985E-01$	$1.2520E-00$
0.01	$7.4128E-04$	$4.7023E-03$	$3.6591E-02$	$2.6558E-01$	$1.1851E-00$
0.001	$7.4128E-07$	$1.4870E-05$	$3.6591E-04$	$8.3985E-03$	$9.98891E-02$

Table 2.6: Maximal absolute error in case 2 where $\alpha = 0.5$ for Example 2.2.

h	$\lvert e(x)\rvert$	$\lvert e^{\alpha}(x)\rvert$	$\lvert e^{2\alpha}(x)\rvert$
0.1	$44.8919E-02$	$12.1447E-01$	$40.8441E-01$
0.01	$44.8919E-05$	$38.4049E-04$	$40.8441E-03$
0.001	$14.1960E-07$	$12.1447E-06$	$40.8441E-05$
h	$\lvert e^{3\alpha}(x)\rvert$	$\lvert e^{4\alpha}(x)\rvert$	$\lvert e^{5\alpha}(x)\rvert$
0.1	$16.6379E-01$	$27.1421E-01$	$76.0656E-01$
0.01	$52.6138E-03$	$27.1421E-02$	$24.0540E-01$
0.001	$16.6379E-04$	$27.1421E-03$	$76.0656E-02$

Table 2.7: Maximal absolute errors in case 1 where $\alpha = 0.8$ for Example 2.2.

h	$\lvert e(x)\rvert$	$\lvert e^{\alpha}(x)\rvert$	$\lvert e^{2\alpha}(x)\rvert$	$\lvert e^{3\alpha}(x)\rvert$	$\lvert e^{4\alpha}(x)\rvert$
0.1	$1.6113E-04$	$2.8349E-03$	$6.1216E-02$	$1.7342E-01$	$1.9889E-01$
0.01	$9.1158E-08$	$5.8825E-07$	$2.8102E-05$	$2.5922E-03$	$1.3473E-02$
0.001	$9.8342E-12$	$4.1473E-10$	$1.8135E-07$	$5.3441E-05$	$9.9985E-04$

Table 2.8: Maximal absolute errors in case 2 where $\alpha = 0.8$ for Example 2.2.

h	$\lvert e(x)\rvert$	$\lvert e^{\alpha}(x)\rvert$	$\lvert e^{2\alpha}(x)\rvert$
0.1	$8.3492E-05$	$4.5619E-04$	$7.1201E-03$
0.01	$8.8939E-09$	$1.5931E-08$	$1.1923E-06$
0.001	$1.1242E-12$	$4.2113E-10$	$8.4152E-08$
h	$\lvert e^{3\alpha}(x)\rvert$	$\lvert e^{4\alpha}(x)\rvert$	$\lvert e^{5\alpha}(x)\rvert$
0.1	$5.6612E-03$	$4.8146E-02$	$2.1197E-01$
0.01	$6.2981E-04$	$2.2229E-03$	$5.9886E-02$
0.001	$6.1347E-07$	$2.4749E-05$	$1.1191E-02$

Example 2.3 Consider the linear fractional differential equation

$$D^{4\alpha}y(x) + D^{\alpha}y(x) + y(x) = g(x), \quad 0 < \alpha \leq 1, \tag{2.28}$$

where

$$g(x) = \frac{x^4}{24} + x^{4-\alpha}\left(\frac{1}{\Gamma(5-\alpha)} + \frac{x^{-3\alpha}}{\Gamma(5-4\alpha)}\right),$$

subject to

$$y(0) = y(1) = 0.$$

The exact solution of this problem is

$$y(x) = x^4/24.$$

All error bounds obtained, for $\alpha = 0.5, 0.8$, and for $0 \leq x \leq 1$ are shown in Tables 2.9–2.10, to illustrate the accuracy of the spline method of polynomial form. We have shown the maximal error's values in each case. Note that $|e^{m\alpha}(x)| = |D^{m\alpha}S_k(x) - D^{m\alpha}y(x)|$, for $m = 0(1)4$.

Table 2.9: Error bounds for Example 2.3 when $\alpha = 0.5$.

h	$\lvert e(x)\rvert$	$\lvert e^{\alpha}(x)\rvert$	$\lvert e^{2\alpha}(x)\rvert$	$\lvert e^{3\alpha}(x)\rvert$	$\lvert e^{4\alpha}(x)\rvert$
0.1	$2.1751E-04$	$2.6173E-03$	$4.9918E-02$	$6.9235E-02$	$1.9969E-01$
0.01	$1.1009E-07$	$1.9996E-07$	$8.3841E-05$	$5.1127E-03$	$9.6114E-02$
0.001	$9.9890E-12$	$2.9838E-09$	$8.9655E-07$	$9.0180E-05$	$1.2423E-02$

Table 2.10: Error bounds for Example 2.3 when $\alpha = 0.8$.

h	$\lvert e(x)\rvert$	$\lvert e^{\alpha}(x)\rvert$	$\lvert e^{2\alpha}(x)\rvert$	$\lvert e^{3\alpha}(x)\rvert$	$\lvert e^{4\alpha}(x)\rvert$
0.1	$1.2638E-04$	$1.5003E-03$	$1.3115E-02$	$6.8465E-02$	$1.7016E-01$
0.01	$1.2638E-08$	$9.4667E-07$	$5.2212E-05$	$1.7197E-03$	$2.6969E-02$
0.001	$1.2638E-12$	$5.9731E-10$	$2.0786E-07$	$4.3198E-05$	$4.2743E-03$

Chapter Three

Analysis of Fractional Spline Interpolation

Chapter 3

Analysis of Fractional Spline Interpolation

3.1 Introduction

As shown in chapter 2, fractional differential equations have been the focus of many studies due to their frequent appearances in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering. Most fractional differential equations do not have exact analytic solutions, so approximation and numerical techniques must be used. The fractional spline function of a polynomial form (see [**?**, **?**, 31, 58]) is a new approach to provide an analytical approximation to linear and nonlinear problems, and it is particularly valuable as a tool for scientists and applied mathematicians, because they provide immediate and visible symbolic terms of numerical approximate solutions to both linear and nonlinear differential equations.

In this chapter, we construct a new fractional spline which interpolates the $\Big(\frac{1}{2}$-th derivative for the first case, $\frac{1}{2}$, $\frac{3}{2}$-th derivatives for the second case, and $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$-th derivatives for the last case $\Big)$ of a given function at the knots and its value at the beginning of the interval is considered. We obtain a direct simple formula for the proposed fractional spline.

3.2 Analysis of Fractional Splines Interpolation–$\left(0, \frac{1}{2}\right)$ Case and Optimal Error bounds

Here, we construct a class of interpolating fractional splines of degree $j\alpha$, for $j = 2, 4, 6$; $\alpha = 0.5$ and error estimates for these splines are also represented. Since all cases considered are similar, details are given only for the first case of 2α.

Let $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$ be a uniform partition of $[0, 1]$. Set the stepsize $h = x_{i+1} - x_i$ $(i = 0(1)n - 1)$. Since all cases considered are similar, details are given only for the first case. We study the following new cases:

3.2.1 Spline of Degree 2α (Existence and Uniqueness)

We suppose that $s^{(1/2)}(x) \in C^2[0, 1]$ and $s(x)$ in each subinterval $[x_i, x_{i+1}]$ has a form:

$$s(x) = a_i(x - x_i) + b_i(x - x_i)^{1/2} + c_i, \tag{3.1}$$

where a_i, b_i, and c_i are constants to be determined.

Theorem 3.1. *Suppose that $s^{(1/2)}(x) \in C^2[0, 1]$ and $s(x)$ in each subinterval $[x_i, x_{i+1}]$ has the form (3.1). Then there exist a unique $s(x)$ such that*

$$\begin{aligned} s_i^{(1/2)} &= f_i^{(1/2)}, \quad (i = 0(1)n), \\ s_0 &= f_0. \end{aligned} \tag{3.2}$$

The fractional spline which satisfies (3.2) in $[x_i, x_{i+1}]$ is of the form:

$$s(x) = s_i A_0(t) + s_{i+1} A_1(t) + h^{1/2} s_i^{(1/2)} A_2(t), \tag{3.3}$$

where

$$A_0(t) = 1 - t^{3/2}, \quad A_1(t) = t^{3/2}, \quad A_2(t) = \frac{2}{\sqrt{\pi}}\left(t^{1/2} - t^{3/2}\right), \tag{3.4}$$

and $x = x_i + th$, $t \in [0, 1]$, *with a similar expression for* $s(x)$ *in* $[x_{i-1}, x_i]$.

The coefficient s_i *in* (3.3) *are given by the recurrence formula:*

$$\begin{aligned} s_i &= s_{i-1} - \frac{2}{3\sqrt{\pi}} h^{\frac{1}{2}} \left(f_{i-1}^{(1/2)} + 2 f_i^{(1/2)} \right), \\ s_0 &= f_0, \quad i = 1(1)n. \end{aligned} \tag{3.5}$$

Proof. Indeed we can express $p(t)$ in $[0, 1]$ in the following form:

$$p(t) = p_0 A_0(t) + p_1 A_1(t) + p_0^{(1/2)} A_2(t).$$

To determine A_0, A_1, A_2, we write the above equality for $p(t) = 1, t^{1/2}, t^{3/2}$ we get

$$\begin{aligned} A_0 + A_1 &= 1, \\ A_1 + \frac{\sqrt{\pi}}{2} A_2 &= t^{1/2}, \\ A_1 &= t^{3/2}. \end{aligned}$$

Solving this, we obtain

$$A_0(t) = 1 - t^{3/2}, \quad A_1(t) = t^{3/2}, \text{ and } \quad A_2(t) = \frac{2}{\sqrt{\pi}} \left(t^{1/2} - t^{3/2} \right).$$

Now, for a fixed $i \in \{0, 1, ..., n-1\}$, set $x = x_i + th$, $t \in [0, 1]$. In the subinterval $[x_i, x_{i+1}]$ the fractional spline $s(x)$ satisfying (3.2) is:

$$s(x) = s_i A_0(t) + s_{i+1} A_1(t) + h^{\frac{1}{2}} f_i^{(1/2)} A_2(t).$$

We have a similar expression for $s(x)$ in $[x_{i-1}, x_i]$. From the continuity condition of $s^{(1/2)}(x_i^-) = s^{(1/2)}(x_i^+)$ we arrive the above recurrence formula (3.5). This completes the proof. □

3.2.2 Error Bounds for the Fractional Spline

In this section, the L_∞ error estimates are presented for the above interpolating lacunary fractional spline in the interval $[0, 1]$.

Theorem 3.2. *Suppose that $s(x)$ be the fractional spline defined in Section 3.2.1, $f^{(1/2)} \in C^2[0, 1]$ and that $f'(0) = f''(0) = 0$, then for any $x \in [0, 1]$ we have*

$$|s(x) - f(x)| \leq \frac{h^2}{4\sqrt{\pi}} \left\| f^{(5/2)} \right\|. \tag{3.6}$$

Proof. Since $s^{(1/2)}(x)$ is the Hermite interpolation polynomial of degree 1 matching $f^{(1/2)}$ at $x = x_i$ and x_{i+1}, we have

$$\left| s^{(1/2)}(x) - f^{(1/2)}(x) \right| \leq \frac{h^2}{4.2!} \left\| D^2 D^{1/2} \right\|.$$

By taking $I_{0|x}^{1/2}$ to both sides of the above equation, we get

$$\left| I_{0|x}^{1/2} \left(s^{(1/2)}(x) - f^{(1/2)}(x) \right) \right| \leq I_{0|x}^{1/2} \left(\frac{h^2}{4.2!} \left\| D^2 D^{1/2} f \right\| \right).$$

Hence

$$|(s(x) - s(0) - f(x) + f(0))| \leq \frac{2}{\sqrt{\pi}} x^{1/2} \left(\frac{h^2}{4.2!} \left\| D^2 D^{1/2} f \right\| \right).$$

Since, $s(0) = f(0)$ and $x \in [0, 1]$, then the last equation becomes

$$|s(x) - f(x)| \leq \frac{h^2}{4\sqrt{\pi}} \left\| D^2 D^{1/2} f \right\|,$$

and since $f'(0) = f''(0) = 0$, following [26], p. 20, we have

$$D^2 D^{1/2} f = D^{5/2} f = f^{(5/2)},$$

which leads to

$$|s(x) - f(x)| \leq \frac{h^2}{4\sqrt{\pi}} \left\| f^{(5/2)} \right\|.$$

Thus we have proved the theorem. □

3.2.3 Spline of Degree 4α Case (Existence and Uniqueness)

We suppose here that $s^{(1/2)}(x) \in C^4[0,1]$ and $s(x)$ in each subinterval $[x_i, x_{i+1}]$ has a form:

$$s(x) = a_i(x-x_i)^2 + b_i(x-x_i)^{3/2} + c_i(x-x_i) + d_i(x-x_i)^{1/2} + e_i. \tag{3.7}$$

From which the following theorem can be obtained:

Theorem 3.3. *Let $s(x)$ be the fractional spline defined in Section 3.2.3. Then there exist a unique $s(x)$ such that*

$$\begin{aligned} &s_i^{(1/2)} = f_i^{(1/2)},\ s_i^{(3/2)} = f_i^{(3/2)}, \quad \textit{for } (i = 0(1)n), \\ &s_0 = f_0. \end{aligned} \tag{3.8}$$

The fractional spline which satisfies (3.8) in $[x_i, x_{i+1}]$ is of the form:

$$s(x) = s_iA_0(t) + s_{i+1}A_1(t) + h^{\frac{1}{2}}\left[f_i^{(1/2)}A_2(t) + f_{i+1}^{(1/2)}A_3(t)\right] + h^{\frac{3}{2}}f_i^{(3/2)}A_4(t), \tag{3.9}$$

where

$$\begin{aligned} &A_0(t) = \frac{1}{5}\left(2t^{7/2} - 7t^{5/2}\right) + 1,\ A_1(t) = -\frac{1}{5}t^{5/2}(2t-7),\ A_2(t) = \frac{2}{75\sqrt{\pi}}t^{1/2}(14t^3 - 89t^2 + 75) \\ &A_3(t) = \frac{32}{75\sqrt{\pi}}t^{5/2}(t-1),\ A_4(t) = \frac{4}{75\sqrt{\pi}}t^{3/2}(2t-25)(t-1), \end{aligned} \tag{3.10}$$

and $x = x_i + th$, $t \in [0,1]$, with a similar expression for $s(x)$ in $[x_{i-1}, x_i]$.

The coefficient s_i in (3.9) are given by the recurrence formula:

$$\begin{aligned} &\frac{21}{16}\sqrt{\pi}(s_i - s_{i-1}) = h^{\frac{1}{2}}\left[\frac{129}{40}f_{i-1}^{(1/2)} + \frac{3}{5}f_i^{(1/2)}\right] + h^{\frac{3}{2}}\left[f^{(3/2)}f_i - \frac{27}{20}f_{i-1}^{(3/2)}\right], \\ &s_0 = f_0, \quad 1 = 1(1)n. \end{aligned} \tag{3.11}$$

Proof. In this case we can express any $p(t)$ in $[0,1]$ in the following form:

$$p(t) = p_0 A_0(t) + p_1 A_1(t) + p_0^{(1/2)} A_2(t) + p_1^{(1/2)} A_3(t) + p_0^{(3/2)} A_4(t)$$

and to determine the coefficients A_j, $j = 0(1)4$, we write the above equality for $p(t) = 1, t^{1/2}, t^{3/2}, t^{5/2}, t^{7/2}$.

By the same technique of Theorem 3.1 we obtain the desired result and consequently the proof is completed. □

3.2.4 Error Bounds for the Fractional Spline of Degree 4α Case

Here, we will derive the L_∞ error estimates for the fractional spline that we have mentioned in Section 3.2.3, the error bounds have shown in the below theorem and its proof is similar subsequence of Theorem 3.2.

Theorem 3.4. *Suppose that $s(x)$ be the fractional spline defined in Section 3.2.3, $f^{(1/2)} \in C^4[0,1]$ and that $f^{(p)}(0) = 0,\ p = 1,2,3,4,$ then for any $x \in [0,1]$ we have*

$$|s(x) - f(x)| \le \frac{h^4}{(8)(4!)\sqrt{\pi}} \|f^{(9/2)}\|. \tag{3.12}$$

Proof. Because $s^{(1/2)}(x)$ is the Hermite interpolation polynomial of degree 3 matching $f^{(1/2)}$, $f^{(3/2)}$ at $x = x_i$ and x_{i+1}, we have

$$|s(x) - f(x)| \le \frac{h^4}{(8)(4!)\sqrt{\pi}} \left\| D^4 D^{1/2} f \right\|,$$

and following [26], p. 20, we have

$$D^4 D^{1/2} f = D^{9/2} f = f^{(9/2)},$$

which leads to

$$|s(x) - f(x)| \le \frac{h^4}{(8)(4!)\sqrt{\pi}} \left\| f^{(9/2)} \right\|,$$

which proves the theorem. □

3.2.5 Spline of Degree 6α Case (Existence and Uniqueness)

We suppose here that $s^{(1/2)}(x) \in C^6[0,1]$ and $s(x)$ in each subinterval $[x_i, x_{i+1}]$ has a form:

$$s(x) = a_i(x-x_i)^3 + b_i(x-x_i)^{5/2} + c_i(x-x_i)^2 + d_i(x-x_i)^{3/2} + e_i(x-x_i) + f_i. \tag{3.13}$$

Which deduces the following theorem:

Theorem 3.5. *Let $s(x)$ be the fractional spline defined in Section 3.2.5. Then there exist a unique $s(x)$ such that*

$$\begin{aligned} &s_i^{(1/2)} = f_i^{(1/2)},\ s_i^{(3/2)} = f_i^{(3/2)},\ s_i^{(5/2)} = f_i^{(5/2)}, \quad \textit{for } (i = 0(1)n), \\ &s_0 = f_0. \end{aligned} \tag{3.14}$$

The fractional spline which satisfies (3.14) in $[x_i, x_{i+1}]$ is of the form:

$$\begin{aligned} s(x) &= s_i A_0(t) + s_{i+1}A_1(t) + h^{\frac{1}{2}}\left[f_i^{(1/2)}A_2(t) + f_{i+1}^{(1/2)}A_3(t)\right] \\ &\quad + h^{\frac{3}{2}}\left[f_i^{(3/2)}A_4(t) + f_{i+1}^{(3/2)}A_5(t)\right] + h^{\frac{5}{2}} f_i^{(5/2)}A_6(t), \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} &A_0(t) = \frac{1}{3}\left(176t^{9/2} - 99t^{7/2} - 80t^{11/2} + 3\right) + 1,\ A_1(t) = \frac{1}{3}t^{7/2}\left(80t^2 - 176t + 99\right), \\ &A_2(t) = -\frac{2}{63\sqrt{\pi}}t^{1/2}(656t^5 - 1520t^4 + 927t^3 - 63),\ A_3(t) = -\frac{256}{63\sqrt{\pi}}t^{7/2}(8t-9)(t-1), \\ &A_4(t) = -\frac{4}{945\sqrt{\pi}}t^{3/2}\left(1360t^4 - 3376t^3 + 2331t^2 - 315\right), \\ &A_5(t) = \frac{256}{945\sqrt{\pi}}t^{7/2}(10t-9)(t-1),\ A_6(t) = -\frac{8}{945\sqrt{\pi}}t^{5/2}(4t-3)(20t-21)(t-1), \end{aligned} \tag{3.16}$$

and $x = x_i + th$, $t \in [0,1]$, with a similar expression for $s(x)$ in $[x_{i-1}, x_i]$.

The coefficient s_i in (3.15) are given by the recurrence formula:

$$\frac{1155}{16}\sqrt{\pi}(s_i - s_{i-1}) = h^{\frac{1}{2}}\left[\frac{355}{8}f_{i-1}^{(1/2)} + 100f_i^{(1/2)}\right] + h^{\frac{3}{2}}\left[\frac{115}{12}f_i^{(3/2)} - \frac{40}{3}f_{i-1}^{(3/2)}\right] + h^{\frac{5}{2}}\left[f_{i-1}^{(5/2)} + \frac{59}{12}f_i^{(5/2)}\right], \tag{3.17}$$

$$s_0 = f_0, \quad i = 1(1)n.$$

Proof. In this case we can express any $p(t)$ in $[0, 1]$ in the following form:

$$p(t) = p_0A_0(t) + p_1A_1(t) + p_0^{(1/2)}A_2(t) + p_1^{(1/2)}A_3(t) + p_0^{(3/2)}A_4(t) + p_1^{(3/2)}A_5(t) + p_0^{(5/2)}A_6(t),$$

and to determine the coefficients A_j, $j = 0(1)6$, we write the above equality for $p(t) = 1, t^{1/2}, t^{3/2}, t^{5/2}, t^{7/2}\, t^{9/2}, t^{11/2}$. By the same technique of Theorem **??** we obtain the desired result and hence the proof is completed. □

3.2.6 Error Bounds for the Fractional Spline of Degree 6α Case

Error estimates for the fractional spline that we have mentioned in Section 3.2.5 are explained by the following theorem:

Theorem 3.6. *Suppose that $s(x)$ be the fractional lacunary spline defined in section 3.2.3, $f^{(1/2)} \in C^6[0, 1]$ and that $f^{(p)}(0) = 0,\ p = 1(1)6$, then for any $x \in [0, 1]$ we have*

$$|s(x) - f(x)| \le \frac{h^6}{(32)(6!)\sqrt{\pi}}\|D^{\frac{13}{2}}f\|. \tag{3.18}$$

Proof. Here, since $s^{(1/2)}(x)$ is the Hermite interpolation polynomial of degree 3 matching $f^{(1/2)}$, $f^{(3/2)}$, $f^{(5/2)}$ at $x = x_i$ and x_{i+1}, we have

$$|s(x) - f(x)| \le \frac{h^6}{(32)(6!)\sqrt{\pi}}\left\|D^6D^{1/2}f\right\|,$$

and following [26], p. 20, we have

$$D^6 D^{1/2} f = D^{13/2} f = f^{(13/2)}.$$

This gives

$$|s(x) - f(x)| \leq \frac{h^6}{(32)(6!)\sqrt{\pi}} \left\| f^{(13/2)} \right\|.$$

This proves the theorem. □

3.2.7 Algorithms

The following steps are needed in solving a problem:

Step 1. The above formulation and analysis was done in $[0, 1]$. However, this does not constitute a serious restriction since the same techniques can be carried out for the general interval $[a, b]$. This is achieved using the linear transformation

$$x = \frac{1}{b-a} t - \frac{a}{b-a} \tag{3.19}$$

from $[a, b]$ to $[0, 1]$.

Step 2. Use the equations (3.5), (3.11) and (3.17) to compute s_i for $i = 1(1)n$, respectively, in each cases.

Step 3. Use the equations (3.3), (3.9) and (3.15) to compute $s(x)$ at n equally spaced points in each subinterval $[x_i, x_{i+1}]$ for $i = 1(1)n - 1$ and in each case.

3.2.8 Illustrations

To illustrate our methods and to compare each of them with the other one, we have solved two examples of fractional equation. We have implemented all calculations with MATLAB 12b.

Example 3.1 Consider the following fractional differential equation

$$f^{(\frac{1}{2})}(x) - \sqrt{\pi}\, x^{\frac{15}{2}} = 0, \quad x \in [0,1], \tag{3.20}$$

with $f(0) = 0$.

For which, all actual error bounds for each case are presented in Table 3.1,

Table 3.1: The observed maximum errors

	Fractional Splines		
h	degree 2α	degree 4α	degree 6α
1/10	$1.218750000000000E-01$	$6.284179687500002E-04$	$4.582214355468752E-07$
1/20	$3.046875000000001E-02$	$3.927612304687501E-05$	$7.159709930419925E-09$
1/40	$7.617187500000002E-03$	$2.454757690429688E-06$	$1.118704676628113E-10$
1/80	$1.904296875000000E-03$	$1.534223556518555E-07$	$1.747976057231427E-12$

Example 3.2 Let $f(x) = x^{\frac{13}{2}}$, $x \in [0,1]$.

For which, all actual error bounds for each case are presented in Table 3.2.

Table 3.2: The observed maximum errors

	Fractional Splines		
h	degree 2α	degree 4α	degree 6α
1/10	$3.454908229515436E-01$	$8.637270573788592E-04$	$1.439545095631432E-07$
1/20	$8.637270573788590E-02$	$5.398294108617870E-05$	$2.249289211924113E-09$
1/40	$2.159317643447148E-02$	$3.373933817886169E-06$	$3.514514393631426E-11$
1/80	$5.398294108617869E-03$	$2.108708636178855E-07$	$5.491428740049104E-13$

Example 3.3 Consider the following fraction differential equation

$$f^{(1/2)}(x) - \frac{40320}{\Gamma(8.5)} x^{\frac{15}{2}} + \frac{5040}{\Gamma(7.5)} x^{\frac{13}{2}} = 0, \quad \text{with } f(0) = 1.5, \quad x \in [0,1]. \tag{3.21}$$

Numerical and exact solutions are presented in Table 3.3, we give here the fractional spline of degree 6α for $h = 0.1$. Also, the exact and numerical solutions are demonstrated in Figure 3.1 for $h = 0.2$.

Table 3.3: Exact, approximate and absolute error

x	Exact Solution	Approximation Solution	Absolute Error
0.0	1.500000000000000	1.500000000000000	0
0.1	1.499999910000000	1.499999954520380	$2.065014825802791E-8$
0.2	1.499989760000000	1.499997977657317	$8.217657316622606E-6$
0.3	1.499846910000000	1.499975003854680	$1.280938546797117E-4$
0.4	1.499016960000000	1.499843000789528	$8.260407895279709E-4$
0.5	1.496093750000000	1.499357612945718	$3.263862945717788E-3$
0.6	1.488802560000000	1.498062216177736	$9.259656177735387E-3$
0.7	1.475293710000000	1.495425272300199	$2.013156230019919E-2$
0.8	1.458056960000000	1.491449436911763	$3.339247691176306E-2$
0.9	1.452170310000000	1.488165647180297	$3.599533718029724E-2$
1.0	1.500000000000000	1.492576230669988	$7.423769330012542E-3$

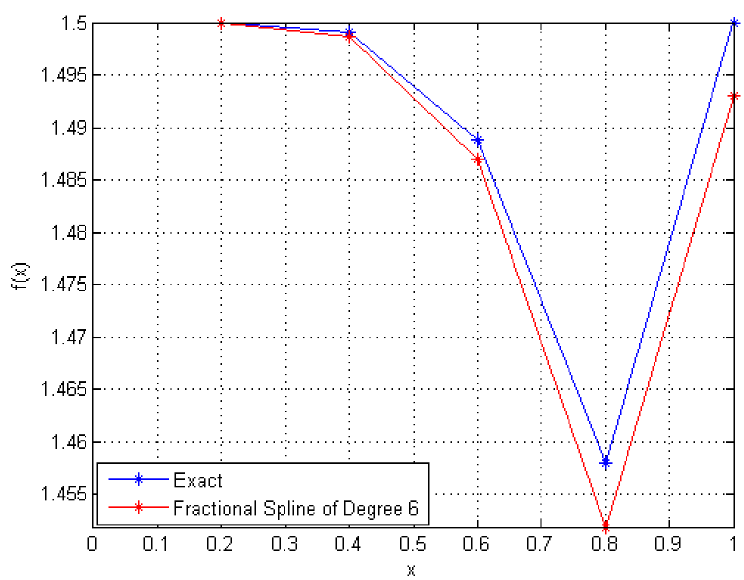


Figure 3.1: Exact and approximate solutions of Example 3.3 with $h = 0.2$.

3.3 General algorithms for the Fractional Spline Approximation Function with Applications

In this section, depends on Subsections 3.2.1, 3.2.3 and 3.2.5 we construct a general formula for fractional spline which interpolates the α-derivatives of a given function at the knots and its value and its 2α- derivative at the beginning of the interval considered. We obtain a direct simple formula for the proposed spline. L_∞ error bounds for the function and its first three derivatives are derived. A direct application of the proposed method is the approximation of the integral function

$$f(x) = I_a^{\alpha}\,{}_cD_a^{\alpha} f(x). \tag{3.22}$$

3.3.1 Existence and Uniqueness

Let $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$ be a uniform partition of $[0, 1]$. We suppose that $s(x), D^{\alpha}s(x)$, and $D^{2\alpha}s(x)$ are continuous on $[0, 1]$ and $s(x)$ in each subinterval $[x_i, x_{i+1}]$ has a form:

$$s(x) = a_i(x - x_i)^{3\alpha} + b_i(x - x_i)^{2\alpha} + c_i(x - x_i)^{\alpha} + d_i. \tag{3.23}$$

Set the stepsize $h = x_{i+1} - x_i\,(i = 0(1)n)$. Then, we prove the following existence and uniqueness theorem:

Theorem 3.7. *Suppose that $s(x), D^{\alpha}s(x)$, and $D^{2\alpha}s(x)$ are continuous on $[0, 1]$ and $s(x)$ in each subinterval $[x_i, x_{i+1}]$ has the form (3.23). Then, there exist a unique $s(x)$ such that*

$$\begin{aligned} D^{\alpha}s_i &= D^{\alpha}f_i, \quad (i = 0(1)n),\\ s_0 = f_0, &\quad D^{2\alpha}s_0 = D^{2\alpha}f_0. \end{aligned} \tag{3.24}$$

The fractional spline which satisfies (3.24) in $[x_i, x_{i+1}]$ is:

$$s(x) = s_iA_0(t) + h^{\alpha}D^{\alpha}f_iA_1(t) + h^{2\alpha}D^{2\alpha}s_iA_2(t) + h^{\alpha}D^{\alpha}f_{i+1}A_3(t), \tag{3.25}$$

where

$$\begin{aligned} &A_0 = 1, \quad A_1 = \frac{1}{\Gamma(\alpha+1)}t^{\alpha} - \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}t^{3\alpha},\\ &A_2 = \frac{1}{\Gamma(2\alpha+1)}t^{2\alpha} - \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha}, \quad A_3 = \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}t^{3\alpha}, \end{aligned} \tag{3.26}$$

and $x = x_i + th$, $t \in [0, 1]$, with a similar expression for $s(x)$ in $[x_{i-1}, x_i]$.

The coefficients s_i and $D^{2\alpha}s_i$ in (3.25) are given by the recurrence formula:

$$\begin{cases} s_i & = s_{i-1} + \left(\frac{1}{\Gamma(\alpha+1)} - \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\right) h^\alpha D^\alpha s_{i-1} + \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)} h^\alpha D^\alpha s_i \\ & \quad + \left(\frac{1}{\Gamma(\alpha+1)} - \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)\Gamma(3\alpha+1)}\right) h^{2\alpha} D^{2\alpha} s_{i-1}, \\ D^{2\alpha} s_i & = \left(1 - \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\right) D^{2\alpha} s_{i-1} + \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} h^{-\alpha} \left(-D^\alpha f_{i-1} + D^\alpha f_i\right) \\ s_0 & = f_0, \ D^{2\alpha} s_0 = D^{2\alpha} f_0; \quad (i = 1(1)n). \end{cases} \tag{3.27}$$

Proof. We can write $P(t)$ in $[0, 1]$ as follows:

$$P(t) = P(0)A_0(t) + D^\alpha P(0)A_1(t) + D^{2\alpha} P(0)A_2(t) + D^\alpha P(0)A_3(t).$$

To determine $A_0(t)$, $A_1(t)$, $A_2(t)$ and $A_3(t)$, we will write the equality for $P(t) = 1$, t^α, $t^{2\alpha}$ and $t^{3\alpha}$ then we arrive the following system:

$$\begin{aligned} 1 &= A_0(t), \\ t^\alpha &= \Gamma(\alpha + 1) \left(A_1(t) + A_3(t)\right), \\ t^{2\alpha} &= \Gamma(2\alpha + 1)A_2(t) + \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + 1)} A_3(t), \\ t^{3\alpha} &= \frac{\Gamma(3\alpha + 1)}{\Gamma(2\alpha + 1)} A_3(t), \end{aligned}$$

these imply that:

$$\begin{aligned} & A_0 = 1, \quad A_1 = \frac{1}{\Gamma(\alpha + 1)} t^\alpha - \frac{\Gamma(2\alpha + 1)}{\Gamma(3\alpha + 1)} t^{3\alpha}, \\ & A_2 = \frac{1}{\Gamma(2\alpha + 1)} t^{2\alpha} - \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + 1)\Gamma(3\alpha + 1)} t^{3\alpha}, \quad A_3 = \frac{\Gamma(2\alpha + 1)}{\Gamma(3\alpha + 1)} t^{3\alpha}. \end{aligned}$$

Now for a fixed $i \in \{0, 1, ..., n - 1\}$, set $x = x_i + th$, $t \in [0, 1]$. In the subinterval $[x_i, x_{i+1}]$ the fractional spline $s(x)$ satisfying (3.24) is:

$$s(x) = s_i A_0(t) + h^\alpha D^\alpha f_i A_1(t) + h^{2\alpha} D^{2\alpha} s_i A_2(t) + h^\alpha D^\alpha f_i A_3(t).$$

We have a similar expression for $s(x)$ in $[x_{i-1}, x_i]$. From the continuity conditions $s(x_i^-) =$

$s(x_i^+)$ and $D^{2\alpha}s(x_i^-) = D^{2\alpha}s(x_i^+)$ we arrive the above recurrence formula (3.27). This completes the proof. □

3.3.2 Error Estimations

In this section, we prove some results on error estimations for fractional spline function given by the L_∞ error estimates are presented for the above interpolating fractional spline and its α, 2α, 3α-th derivatives in $[0, 1]$ and note that $\|\cdot\|$ denotes the L_∞ norm.

Lemma 3.8. *[7] If s and t are positive real numbers, $\{a_i\}_{i=0}^k$ is a sequence satisfying $a_0 \geq \frac{-t}{s}$, and*

$$a_{i+1} \leq (1+s)a_i + t, \quad \text{for each } i = 0(1)k-1,$$

then

$$a_{i+1} \leq e^{(i+1)s}\left(a_0 + \frac{t}{s}\right) - \frac{t}{s}.$$

Lemma 3.9. *Let $s(x)$ be the fractional spline defined in Section 3.3.1. If $D^{m.\alpha}f \in C[0,1]$ $(m = 0(1)4)$ then for $i = 0(1)n$ we have*

$$\left|D^{2\alpha}s_i - D^{2\alpha}f_i\right| \leq \begin{cases} 0, & \text{for } i = 0 \\ k_1 h^{2\alpha}\left\|D^{4\alpha}f\right\|, & \text{for } i = 1 \\ \frac{k_1}{k_2-1}\cdot h^{2\alpha}\left\|D^{4\alpha}f\right\|\left[e^{(k_2-1)i}-1\right], & \text{for } i = 2(1)n \end{cases} \tag{3.28}$$

where

$$k_1 = \left|\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)\Gamma(3\alpha+1)} - \frac{1}{\Gamma(2\alpha+1}\right| \quad \text{and} \quad k_2 = \left|1 - \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2}\right|.$$

Proof. We have $D^{2\alpha}s_0 = D^{2\alpha}f_0$ by (3.24). From (3.27) for $i = 1$ we have by expanding the right-hand side by Taylor's expansions that:

$$\begin{aligned}\left|D^{2\alpha}s_1 - D^{2\alpha}f_1\right| &= \left|\left(1-\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\right)D^{2\alpha}s_0 + \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}h^{-\alpha}\left(-D^{\alpha}f_0 + D^{\alpha}f_1\right) - D^{2\alpha}f_1\right| \\ &\le k_2\left|D^{2\alpha}s_0 - D^{2\alpha}f_0\right| + k_1h^{2\alpha}\left\|D^{4\alpha}f\right\| \\ &= k_1h^{2\alpha}\left\|D^{4\alpha}f\right\|,\end{aligned}$$

since $D^{2\alpha}s_0 = D^{2\alpha}f_0$.

Similarly from (3.27) for $j = 0(1)n-1$, we have

$$\left|D^{2\alpha}s_{j+1} - D^{2\alpha}f_{j+1}\right| \le k_2\left|D^{2\alpha}s_j - D^{2\alpha}f_j\right| + k_1h^{2\alpha}\left\|D^{2\alpha}f\right\|.$$

By using Lemma 3.8 and the fact that $D^{2\alpha}s_0 = D^{2\alpha}f_0$, we have for $i \ge 2$

$$\left|D^{2\alpha}s_i - D^{2\alpha}f_i\right| \le \frac{k_1}{k_2-1}\cdot h^{2\alpha}\left\|D^{4\alpha}f\right\|\left[e^{(k_2-1)i}-1\right].$$

This completes the proof. □

Theorem 3.10. *Let $s(x)$ be the fractional spline defined in Section 3.3.1. If $D^{m.\alpha}f \in C[0,1]$ ($m = 0(1)4$) then for any $x \in [0,1]$ we have*

$$\left|D^{3\alpha}s(x) - D^{3\alpha}f(x)\right| \le \left(\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}\cdot\frac{k_1}{k_2-1}\cdot h^{2\alpha}\left[e^{(k_2-1)i}-1\right] + k_3h^{\alpha}\right)\left\|D^{4\alpha}f\right\|, \tag{3.29}$$

$$\left|D^{2\alpha}s(x) - D^{2\alpha}f(x)\right| \le k_1\left(\frac{k_2}{k_2-1}\cdot\left[e^{(k_2-1)i}-1\right]+1\right)h^{2\alpha}\left\|D^{4\alpha}f\right\|, \tag{3.30}$$

$$\left|D^{\alpha}s(x) - D^{\alpha}f(x)\right| \le \frac{k_1}{\Gamma(\alpha+1)}\cdot\left(\frac{k_2}{k_2-1}\cdot\left[e^{(k_2-1)i}-1\right]+1\right)h^{3\alpha}\left\|D^{4\alpha}f\right\|, \tag{3.31}$$

where k_1 and k_2 are given in Lemma 3.9, and

$$k_3 = \left|\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)} - \frac{1}{\Gamma(\alpha+1}\right|.$$

Proof. Differentiating both sides of (3.25) with respect to x, $(x = x_i + th)$, we get

$$D^{2\alpha}s(x) = \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}t^{\alpha}h^{-\alpha}\left(-D^{\alpha}f_i + D^{\alpha}f_{i+1}\right) + \left(1 - \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2}t^{\alpha}\right)D^{2\alpha}s_i, \tag{3.32}$$

$$D^{3\alpha}s(x) = -\Gamma(2\alpha+1)h^{-2\alpha}D^{\alpha}f_i - \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}h^{-\alpha}t^{\alpha}D^{2\alpha}s_i + \Gamma(2\alpha+1)h^{-2\alpha}D^{\alpha}f_{i+1}. \tag{3.33}$$

Subtracting $D^{3\alpha}f(x)$ from both sides of (3.33) and expanding the right-hand side by Taylor's expansion, we obtain

$$\left|D^{3\alpha}s(x) - D^{3\alpha}f(x)\right| \le \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}\left|D^{2\alpha}s_i - D^{2\alpha}f_i\right| + \left|\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)} - \frac{1}{\Gamma(\alpha+1}\right|h^{\alpha}\left\|D^{4\alpha}\right\|,$$

which, together with (3.28) and $k_3 = \left|\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)} - \frac{1}{\Gamma(\alpha+1}\right|$ lead to (3.29).

Similarly, Subtracting $D^{2\alpha}f(x)$ from both sides of (3.33) and using the same technique, we obtain

$$\left|D^{2\alpha}s(x) - D^{2\alpha}f(x)\right| \le k_2\left|D^{2\alpha}s_i - D^{2\alpha}f_i\right| + k_1h^{2\alpha}\left\|D^{4\alpha}\right\|,$$

which, together with (3.28) give (3.30). Since $D^{\alpha}s_i = D^{\alpha}f_i$, from (3.26), we can write

$$D^{\alpha}s(x) - D^{\alpha}f(x) = I^{\alpha}_{x_i|x}\left[D^{2\alpha}s(x) - D^{2\alpha}f(x)\right].$$

Then using (3.30) we get (3.31). Thus the proof is completed. □

Lemma 3.11. *Let $s(x)$ be the fractional spline defined in Section 3.3.1. If $D^{m.\alpha}f \in C[0,1]$ $(m = 0(1)4)$, then for $i = 0(1)n$ we have*

$$|s_i - f_i| \le \begin{cases} 0, & \text{for } i = 0 \\ k_4h^{4\alpha}\left\|D^{4\alpha}f\right\|, & \text{for } i = 1 \\ k_4 \cdot i \cdot h^{4\alpha}\left\|D^{4\alpha}f\right\|, & \text{for } i = 2(1)n \end{cases} \tag{3.34}$$

where

$$k_4 = \left|\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)^2} - \frac{1}{\Gamma(4\alpha+1)}\right|.$$

Proof. The case $i = 0$ follows from (3.24). From (3.27) for the case $i = 1$ we have by expanding the right-hand side about x_0 using fractional Taylor's expansion,

$$\begin{aligned}|s_1 - f_1| &= \Bigg| s_0 + \left(\frac{1}{\Gamma(\alpha+1)} - \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\right) h^\alpha D^\alpha s_0 + \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)} D^\alpha s_1 \\ &\quad + \left(\frac{1}{\Gamma(\alpha+1)} - \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)\Gamma(3\alpha+1)}\right) h^{2\alpha} D^{2\alpha} s_0 - f_1 \Bigg| \\ &\le |s_0 - f_0| + \left|\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)^2} - \frac{1}{\Gamma(4\alpha+1)}\right| h^{4\alpha} \left\|D^{4\alpha} f\right\| \\ &= k_4 h^{4\alpha} \left\|D^{4\alpha} f\right\|,\end{aligned}$$

since $s_0 = f_0$.

Similarly from (3.27), for $i = 2(1)n$, we get

$$|s_i - f_i| \le |s_{i-1} - f_{i-1}| + k_4 h^{4\alpha} \left\|D^{4\alpha} f\right\|.$$

For a fixed integer i, this inequality implies that

$$|s_i - f_i| \le k_4 \cdot i \cdot h^{4\alpha} \left\|D^{4\alpha} f\right\|.$$

Thus the proof is completed. □

Theorem 3.12. *Let $s(x)$ be the fractional spline defined in Section 3.3.1. If $D^{m.\alpha} f \in C[0,1]$ ($m = 0(1)4$), then for any $x \in [0,1]$ we have*

$$|s(x) - f(x)| \le \left[\frac{k_1^2}{k_2 - 1}\left(e^{(k_2-1)i} - 1\right) + k_4(i+1)\right] h^{4\alpha} \left\|D^{4\alpha} f\right\|. \tag{3.35}$$

Proof. Subtracting $f(x)$ from both sides of (3.25) and applying fractional Taylor's Theorem 1.1 for the right-hand side about x_i, we obtain

$$|s(x) - f(x)| \le |s_i - f_i| + k_1 h^{2\alpha} \left|D^{2\alpha} s_i - D^{2\alpha} f_i\right| + k_4 h^{4\alpha} \left\|D^{4\alpha} f\right\|,$$

which, together with (3.28) lead to (3.35). Thus the proof is completed. □

3.3.3 Algorithms

The following steps are needed in solving a problem:

Step 1. The above formulation and analysis was done in $[0, 1]$. However, this does not constitute a serious restriction since the same techniques can be carried out for the general interval $[a, b]$. This is achieved using the linear transformation

$$x = \frac{1}{b-a}t - \frac{a}{b-a} \tag{3.36}$$

from $[a, b]$ to $[0, 1]$.

Step 2. Use the formulation of (3.27) to compute s_i, s_i', $(i = 1(1)n)$.

Step 3. Use equation (3.25) to compute $s(x)$ at n equally spaced points in each subinterval $[x_i, x_{i+1}]$ $(i = 1(1)n - 1)$.

Step 4. $s^{(1/2)}(x)$, $s'(x)$ and $s^{(3/2)}(x)$ are obtained from $s(x)$.

3.3.4 Numerical Illustrations

In order to demonstrate the efficiency of the proposed method, three numerical examples are considered. All calculations were implemented by MATLAB 12b.

Example 3.4 Consider the fractional differential equation

$$D^{\alpha}y(x) - \frac{2}{\Gamma(3-\alpha)}x^{2-\alpha} = 0, \quad 0 < x \leq 1. \tag{3.37}$$

The exact solution of which is:

$$y(x) = x^2.$$

In this example the approximate and exact solutions are given in the knots x_i, and for which the maximum absolute error is presented for $\alpha = 0.8$ (see Table 3.4).

Table 3.4: Exact, approximate and absolute error

x	Exact Solution	Approximation Solution	Error
0.00	2	2	0
0.10	0.010000000000000	0.009409901887389	$5.900981126109994E-04$
0.20	0.040000000000000	0.040681956533135	$6.819565331349989E-04$
0.30	0.090000000000000	0.098402927569831	$8.402927569831006E-03$
0.40	0.160000000000000	0.185041964592567	$2.504196459256700e-02$
0.50	0.250000000000000	0.302406206975918	$5.240620697591802e-02$
0.60	0.360000000000000	0.451944689961054	$9.194468996105404e-02$
0.70	0.490000000000000	0.634878308687241	$1.448783086872411e-01$
0.80	0.640000000000000	0.852268549612962	$2.122685496129619e-01$
0.90	0.810000000000000	1.105058485208949	$2.950584852089488e-01$
1.00	1.000000000000000	1.394099293010464	$3.940992930104641e-01$

Example 3.5 Let

$$f(t) = t^2 + 1 \text{ in } [1, 2] \tag{3.38}$$

The maximum error bounds for the function and its α, 2α-th derivatives using the proposed method are presented in Table 3.5 in case of $\alpha = 0.5$, and $n = 10,\ 20$ and 100.

Table 3.5: Maximum absolute error for Example 3.5

Step size h	e	$D^{\alpha}e$	$D^{2\alpha}e$
0.10	$1.3176E-02$	0	$2.0113E-02$
0.05	$3.2942E-03$	0	$1.0056E-02$
0.01	$1.3176E-04$	0	$2.0113E-03$

Example 3.6 Consider the following fraction differential equation

$$f^{(1/2)}(x) - \frac{40320}{\Gamma(8.5)}x^{\frac{15}{2}} + \frac{5040}{\Gamma(7.5)}x^{\frac{13}{2}} = 0, \quad \text{with } f(0) = 1.5, \quad x \in [0, 1]. \tag{3.39}$$

Numerical and exact solutions are presented in Table 3.6 using the proposed fractional spline for $\alpha = 0.8$ and $h = 0.1$. Also, the exact and numerical solutions are demonstrated for $\alpha = 0.8$ and $h = 0.25$ in Figure 3.2.

Table 3.6: Exact, approximate and absolute error

x	Exact Solution	Approximation Solution	Absolute Error
0.0	1.500000000000000	1.500000000000000	0
0.1	1.499999910000000	1.499999795455770	$1.145442301009325E-7$
0.2	1.499989760000000	1.499985929268890	$3.830731109655261E-6$
0.3	1.499846910000000	1.499801259288805	$4.565071119522202E-5$
0.4	1.499016960000000	1.498641944946279	$3.750150537213948E-4$
0.5	1.496093750000000	1.494200025943200	$1.893724056799551E-3$
0.6	1.488802560000000	1.482260826829597	$6.541733170402742E-3$
0.7	1.475293710000000	1.458423518407489	$1.687019159251135E-2$
0.8	1.458056960000000	1.424422576829834	$3.363438317016598E-2$
0.9	1.452170310000000	1.402303366778089	$4.986694322191143E-2$
1.0	1.500000000000000	1.460464635470158	$3.953536452984197E-2$

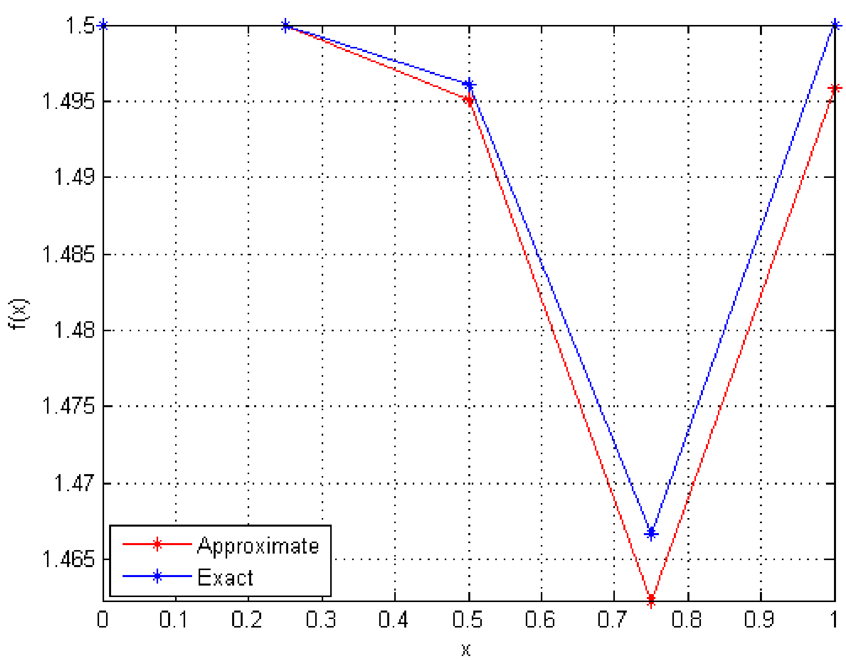


Figure 3.2: Exact and approximate solutions of Example 3.6 with $h = 0.25$.

Chapter Four

Conclusions and Future Works

Chapter 4

Conclusions and Future Works

4.1 Conclusions

In this thesis, we have discussed and constructed different types of fractional lacunary interpolation data by spline functions, which are used in order to improve the solution of fractional differential equations. Furthermore, new results are stated and proved.

In Chapter 2, we introduced a new kind of the fractional spline of polynomial form to be applicable for the case $0 < \alpha \leq 1$. The method is tested by considering two test problems for two fractional ordinary differential equations. From Tables 2.1–2.10, we conclude that the method is suitable for solving FDEs and the error is decrease when h and α are decrease.

In Chapter 3, the existence and uniqueness of three fractional splines of degree $m\alpha$, $m = 2, 4, 6$, $\alpha = 0.5$ are derived and in each case we have obtained direct simple formulas. These formulas are agreeable with those obtained for degree of integer, such as in [43], where a different approach was used. Moreover, in Section 3.3, a new technique using fractional spline function approximation is presented that fits the α-th derivatives at the knots together with the value of the function and its 2α-th derivative at the beginning of the interval, thus obtaining direct simple formulae (3.25) and (3.27). These formulas agree with those obtained for integer previously, such as in [43]. Also, error estimates are derived, which, with the numerical examples, show the method to be efficient. In addition, we conclude that the the error is decrease when h and α are decrease.

4.2 Future Works

The research presented in the thesis focuses on the use of polynomial fractional spline functions to obtain numerical solution of BVPs in FDEs. This investigation has spawned a number of open research problems. We will figure out some of them, and further investigation in specified directions will certainly lead to the improvement and generalization of the exiting algorithms designed for the numerical solution of initial and BVPs in fractional differential equations.

(a) The use of polynomial and non-polynomial fractional spline functions for the solution of BVPs can be extended to higher-order as well as special linear and nonlinear BVPs. Such types of problems have variety of applications in science and engineering.

(b) Polynomial and non-polynomial fractional spline functions can be used for the solution of singular BVPs of fractional order.

(c) The work done so far on the application of polynomial fractional spline functions in developing algorithms for the initial-value problems can be replaced by their counter parts non-polynomial fractional spline functions to improve the accuracy and generalize the algorithms.

(d) The literature on the use of polynomial and non-polynomial fractional splines for the numerical solution of partial differential equations are very limited. Hence this is another area open for future investigations.

Bibliography

Bibliography

[1] Agarwal, R., A propos d'unc note d'une not de M.Pierre Humbert, *C. R. Se'ances Acad. Sci.*, **236** (21) (1953), 2031-2032.

[2] Ahlberg, J. H., Nilson, E. N. and Walsh, J. L., *The Theory of Splines and their applications*, Academic press, 1967.

[3] Andrews, G. E., Askey, R. and Roy, R., *Special Functions*, Cambridge University Press, Cambridge, 1999.

[4] Apostol, T., *Calculus*, Blaisdell Publishing, Waltham, Massachusetts, 1990.

[5] Barkari, E., Metzler, R. and Klafter, J., From continuous time random walks to the fractional Fokker-Planck equation, *Phys. Rev. E*, **61** (1) (2000), 132-138.

[6] Balazs, J. and Turan P., Notes on interpolation. II, III, IV, *Acta Math. Acud. Sci.Hungar*, **8** (1957), 201-215; **9** (1958), 195-214, 243-258.

[7] Burden, R. and Faires, J., *Numerical Analysis*, Brooks/Cole, Pacific Grove, CA, 9th. edition, 2011.

[8] Caponetto, R., Dongola, G., Fortuna, L. and Petráš I., *Fractional Order Systems: Modeling and Control Applications*, World Scientific, Singapore, 2010.

[9] Chang, Y. and Corliss, G., ATOMFT: solving ODE's and DAE's using Taylor Series, *Computers Math. Applic.*, **28** (1994), 209-233.

[10] Chen, Q. and Wang, G., A class of Bézier-like curves, *Comput. Aided Geom. Design*, **20** (2003), 29-39.

[11]

[12] Conte, S. and de Boor, C., *Elementary Numerical Analysis*, 3rd. edition, McGaw-Hill Kogakusha, Tokyo, 1980.

[13] De Boor, C., *A Practical Guide to Splines*, Springer, New York, 1978.

[14] Diethelm, K., An algorithm for the numerical solution of differential equations of fractional order, *Electronic Transactions on Numerical Analysis*, **5** (1997), 1-6.

[15] Djrbashian, M., *Harmonic Analysis and Boundary Value problems in the Complex Domain*, Birkhäuser Verlag, Basel, 1993.

[16] El-Ajou, A., Abu Arqub, O., Al Zhour, Z. and Momani, S., New Results on Fractional Power Series: Theories and Applications, *Entropy*, **15** (2013), 5305-5323.

[17] Fawzy, T. and Holall, F., Notes on Lacunary Interpolation with Splines IV. $(0,2)$ Interpolation with Splines of Degree 6, *J. Approx. Theory*, **49** (1987), 110-114.

[18] Galeone, L. and Garrappa, R., Fractional Adams-Moulton methods, *Math. Comp. simulation*, **79** (2008), 1358-1367.

[19] Hall, M. and Barrick, T., From diffusion-weighted MRI to anomalous diffusion imaging, *Magn. Reson. Med*, **59** (2008), 447-455.

[20] Hamasalh, F., *Investigation in Lacunary Interpolation with Applications*, PhD Thesis, University of Sulaimani, Sulaimani-Kurdistan Region, Iraq, 2009.

[21] Hamasalh, F., Applied lacunary interpolation for solving Boundary value problems, *J. of Modern Engineering Research*, **2** (2) (2012), 118-123.

[22] Hardy, G., Riemann's form of Taylor's series, *J. London Math. Soc.*, **20** (1945), 48-57.

[23] Henrici, P., *Discrete variable methods in ordinary differential equations*, John Wiley, New York, 1962.

[24] Herrmann, R., *Fractional calculus: an introduction for physicists*, GigaHedron, Germany, 2nd edition, 2014.

[25] Ibrahim, R. and Momani, S., On the existence and uniqueness of solutions of a class of fractional differential equations, *J. of Mathematical Analysis and Applications*, **334** (1) (2007), 1-10.

[26] Ishteva, M. K., *Properties and applications of the Caputo fractional operator*, Msc. Thesis, Dept. of Math., Universität Karlsruhe (TH), Sofia, Bulgaria, 2005.

[27] Junsheng, D., Jianye, A. and Mingyu, X., Solution of system of fractional differential equations by Adomian decomposition method, *Appl. Math. Chinese Univ. Ser. B.*, **22** (2007), 17-12.

[28] Li M., Ren J. and Zhu, T., *Series expansion in fractional calculus and fractional differential equations*, Institute of Theoretical Physics, Lanzhou University, Lanzhou, 730000, China, 2001.

[29] Loverro, A., *Fractional Calculus: History, Definitions and Applications for the Engineer*, Dep. of Aerospace and Mechanical Engineering, Univ. of Notre Dame, Notre Dame, IN 46556, U.S.A., 2004.

[30] Magin R., *Fractional Calculus in Bioengineering*, Begell House Publishers, 2006.

[31] Micula, G., Fawzy, T. and Ramadan, Z., A polynomial spline approximation method for solving system of ordinary differential equations, *Babes-Bolyai Cluj-Napoca. Mathematica*, **32** (4) (1987), 55-60.

[32] Mittag-Leffler, M., Sur la nouvelle function $E_{\alpha(x)}$, *Comptes Rendus Acad. Sci. Paris*, **137** (1903), 554-558.

[33] Momani, S., Analytical approximate solution for fractional heat-like and wave-like equations with variable coefficients using the decomposition method, *Applied Mathematics and Computation*, **165** (2) (2005), 459-472.

[34] Momani, S., Odibat, Z. and Alawneh, A., Variational iteration method for solving the space- and time-fractional KdV equation, *Numerical Methods for Partial Differential Equations*, **24** (1) (2008), 262-271.

[35] Monje, C., Chen, Y., Vinagre, B., Xue, D. and Feliu, V., *Fractional-order Systems and Controls, Series: Advances in Industrial Control*, Springer, 2010.

[36] Odibat, Z., and Momani, S., The variational iteration method: an efficient scheme for handling fractional partial differential equations in fluid mechanics, *Computers and Mathematics with Applications. An International Journal*, **58** (11-12) (2009), 2199-2208.

[37] Oldham, K. and Spanier, J., *The Fractional Calculus*, Academic Press, New York. NY, USA, 1974.

[38] Oldham, K. and Spanier, J., *The Fractional Calculus*, Academic Press, New York. NY, USA, 1974.

[39] Odibat, Z. and Shawagfeh, N., Generalized Taylor's formula, *Appl. Math. Comput.*, **186** (2007), 286-293.

[40] Osler, T., *Leibniz rule, the chain rule and Taylor's theorem for fractional derivatives*, Doctoral thesis, New York University, New York, 1970.

[41] Oustaloup, A., *La Derivation Non Entiere: Theorie, Synthese et Applications*, Hermes, Paris, 1995.

[42] Parkash, K., Om, P., Askey, R. and Roy, R., *Topics In Advanced Calculus*, Firewall Media, New Delhi, India, 2008.

[43] Phythian, J. and Williams, R., Direct cubic spline approximation to integrals, *Int. j . numer. methods eng.*, **23** (1986), 305-315.

[44] Podlubny, I., *Fractional Differentional Equations*, Academic Press, San Diego, 1999.

[45] Prenter, P., *Splines and Variational Methods*, Wiley-Interscience, New York, 1975.

[46] Ramadan, M., Spline solutions of first order delay differential equations, *Journal of the Egyptian Mathematical Society*, **13** (1) (2005), 7-18.

[47] Ross, B., *Fractional Calculus and Its Applications*, Proceedings of the Int. Conf. held at the University of New Haven, June 1974 (Lecture Notes in Mathematics), 1975.

[48] Samko, S., Kilbas, A. and Marichev, O., *Fractional Integrals and Derivatives-Theory and Applications, Gordon and Breach Science*, Amsterdam, 1993.

[49] Saxena, R. and Tripathi, H., (0, 2, 3) and (0, 1, 3) Interpolation Through Splines, *Acta Math. Hung.*, **50** (1987), 63–69.

[50] Saxena, A., (0, 1, 2, 4) Interpolation by G-splines, *Acta Math. Hung.*, **51** (1988), 261-271.

[51] Truiljo, J., Rivero, M. and Bonilla, B., On a Riemann-Liouvill Generalize Taylor's Formula, *J. Math. Anal.*, **231** (1999), 255-265.

[52] Usero, D., *Fractional Taylor Series for Caputo Fractional Derivatives. Construction of Numerical Schemes*, Dpto. de Matemática Aplicada, Universidad Complutense de Madrid, Spain, 2008.

[53] Wang, J. and Zhou, Y., Existence and controllability results for fractional semilinear differential inclusions, *Nonlinear Anal. RWA*, **12** (2011), 3642-3653.

[54] Wang, J. and Zhou, Y., A class of fractional evolution equations and optimal controls, *Nonlinear Anal. RWA*, **12** (2011), 262-272.

[55] Wang, J., Zhou, Y. and Wei, W., Fractional Schrodinger equations with potential and optimal controls, *Nonlinear Anal. RWA*, **13** (2012), 2755-2766.

[56] Ward, C. and David, K., *Numerical Mathematics and Computing*, Brooks/Cole Publishing, Pacific Grove, CA, 7th. edition, 2012.

[57] Yuste, S., Acedo, L. and Lindenberg, K., Reaction front in an A+BC reaction-subdiffusion process, *Phys. Rev. E*, **69** (3) (2004), 036-126.

[58] Zahra, W. and Elkholy, S., Quadratic spline solution for boundary value problem of fractional order, *Numer Algor*, **59** (2012), 373-391.